CHROMATIC POLYNOMIALS

BY

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INTRODUCTION

1. Relation of the present work to previous researches on map-coloring and summary of results. The classical unsolved problem with regard to the coloring of maps is to decide rigorously whether or not four colors always suffice for the coloring of any map on a sphere(1). This problem has led to two quite different types of investigation. The characteristics of these two types may be roughly described as follows:

Type 1. Here the emphasis is qualitative, not quantitative. One is content to prove that the class of maps under consideration can be colored, without being primarily interested in the number of ways this can be done. Moreover, since the outstanding problem is the *four*-color problem, the number of colors considered is limited to four. Perhaps the most characteristic method of Type 1 involves the use of the so-called Kempe chains, first introduced by Kempe in an erroneous solution of the four-color problem in 1879 (Kempe [1](2)). The method was revived by Birkhoff in 1912 (cf. Birkhoff [1]) and led through the efforts of Franklin, Reynolds, Winn and others to considerable success. Winn has proved, for instance, that every map of not more than 35 regions can be colored in four colors. Perhaps even more remark-

(1) For a definition of what is meant by "coloring a map," cf. §2 below. Other terms used in this introductory section will also not be defined until later.

(2) Numbers in brackets refer to the bibliography at the end of the paper.

able is his result to the effect that every map containing at most one region of more than six sides can be colored in four colors (cf. Franklin [1, 2], Reynolds [1, 2], and Winn [1, 2, 3]).

Type 2. Here the emphasis is quantitative. Moreover no restriction is made on the number of colors considered. This point of view leads inevitably to certain polynomials each one of which gives the exact number of ways an associated map may be colored in any number of colors. The theory of these so-called "chromatic" polynomials was initiated by Birkhoff in 1912 (Birkhoff [2]) and has been further developed both by him and by Whitney (Birkhoff [3, 4]; Whitney [1, 2]). These researches have not been as successful as the researches of Type 1 in yielding results that are directly connected with the four-color problem. It is certain that the greater generality of the problem here considered has introduced complications which have *so far* rendered the solution of the classical four-color problem more remote by the methods characteristic of Type 2 than it is by the methods of Type 1. Nevertheless a theorem which can be regarded as a weaker form, or a particular case, of a stronger, or more general, theorem is often harder, rather than easier, to prove than the stronger theorem. This is particularly true of theorems proved by mathematical induction, a method especially suited to combinatorial topology. In mathematical induction, a strengthening of the conclusion of the theorem means also a strengthening of the inductive hypothesis. When the proper balance between conclusion and inductive hypothesis has been reached, the proof goes through; otherwise not.

It is hoped that the more general point of view characteristic of Type 2 may lead to a stronger conjecture than the four-color conjecture, which may eventually turn out to be easier to establish. We hazard such a conjecture in §2, Chapter IV. It is also hoped that the theory of the chromatic polynomials may be developed to the point where advanced analytic function theory may be profitably applied.

This paper belongs primarily to Type 2. Its primary object is the study of the chromatic polynomials. Nevertheless, in the later chapters (V and VI) the most characteristic method of Type 1, namely, that of the Kempe chains, has been taken over and modified so as to yield quantitative results in any number of colors. Simultaneously, on the other hand, we have gained by an alternative method a deeper insight into the nature of the results previously obtained only by investigations of Type 1 by use of the Kempe chains. This is true to the extent that we are now able, without using Kempe chains, to prove the reducibility of the following configurations which are fundamental in investigations of Type 1:

1. The four-ring (Birkhoff [1, p. 120]).
2. The five-ring surrounding more than a single region (Birkhoff [1, pp. 120–122]).
3. Four pentagons abutting a single boundary (Birkhoff [1, p. 126]).

4. A boundary of a hexagon abutting three pentagons (Franklin [1, p. 229]).

Undoubtedly numerous other similar configurations can be proved to be reducible by the same methods, which are characteristic of the quantitative point of view taken by investigations of Type 2. Thus, in accordance with Franklin (cf. Franklin [1]), it would probably be possible, without the use of Kempe chains, to prove that any map with fewer than 25 regions can be colored in four colors. Only one further configuration, reducible by Kempe chains, is needed to accomplish this result. Thus the present work can to some extent be regarded as an attempt to bridge the gap between two previously separated points of view.

The earlier chapters are more exclusively of Type 2. The main results of the first chapter are Theorems I and II of §4, concerning the formulas for the reduction(3) of an m-sided region, and their corollary, Theorem 1 of §6, which is used later in an analysis of the theory of the Kempe chains. The second chapter contains the outlines and results of a very extensive calculation of numerous chromatic polynomials. The object of this calculation is the collection of experimental data. It is the basis of our conjecture of Chapter IV, previously referred to. In Chapters III and IV are proved numerous inequalities satisfied by the coefficients of the chromatic polynomials. Chapter III also contains a determinant formula for the chromatic polynomials, different from, but somewhat similar to, the one given by Birkhoff in 1912 (Birkhoff [2]). Both topics treated in Chapter III have close contact with Whitney's notable theorem (Whitney [3]) to the effect that under certain conditions it is possible to draw a simple closed curve passing once and only once through each region but passing through no vertex.

2. **Definitions.** The formal definitions here listed deal with very simple concepts in a terminology which, with a few notable exceptions, is quite conventional. It is therefore suggested that this section be read very rapidly. It may then later be used for purposes of reference as the need may arise.

The term *region* will be used to denote a two-dimensional open point set whose boundary consists of a finite number of analytic arcs and which is homeomorphic with the interior of a circle, or, more generally, homeomorphic with any plane bounded connected open set S whose boundary consists of a finite number of circles without a common point. In the former case the region is said to be *simply connected*. In the latter case the *multiplicity* of *connectivity* is the number of circles in the complete boundary of S.

The term *map* is used in a somewhat more general sense than is customary. We use the term *proper map* when it is necessary to distinguish the usual sense of the word from the more general sense, defined as follows: A map is a collection of regions, finite in number, together with their boundaries, which

(3) The word "reduction" is here used in quite a different connection from the word "reducible" of the previous paragraphs. Both words will, of course, be explained in the sequel.

cover just once the entire area of a closed surface. We shall be concerned exclusively with the case when this surface is a sphere or, what is topologically the same thing, a complete plane closed by the addition of one point at infinity. By a *boundary point of the map* we mean a point on the boundary of at least one of the regions of the map. If a map contains at least one boundary point which lies on the boundary of only one region, the map is called a *pseudo-map*. Otherwise, it is called a *proper map*. In the literature, a *pseudo-map* is sometimes referred to as a map with an isthmus. Such a map always possesses a region which may be described colloquially as touching itself across a boundary. Sometimes, for the sake of brevity, when no confusion can result, a proper map will be called simply a map.

It is clear that the set of all boundary points of a map can be regarded in more than one way as a one-dimensional complex. Those 0-cells of such a complex which are the end points of three or more 1-cells or of just one 1-cell are, however, uniquely determined for a given map. They will be termed *vertices* of the *map* and also *vertices* of the *regions* on whose boundaries they lie. Those 0-cells which are the end points of just two 1-cells are given no special name, as they are not uniquely determined for a given map; and indeed the only reason for introducing them at all is that in certain rather special and uninteresting cases the configuration would not be a 1-dimensional complex. We wish, for instance, to avoid the possibility of a 1-cell having both end points at a single vertex.

The *multiplicity* of a *vertex* is defined as the number of 1-cells of which it is the end point. It is well known that the four-color problem can be reduced to the case when all vertices are of multiplicity 3. Hence, with one notable exception, most of our work will be concerned with maps containing only *triple vertices*, that is, vertices of multiplicity 3. A vertex of multiplicity 1 is called a *free vertex*. It can occur only in a pseudo-map, but not all pseudo-maps contain free vertices.

A maximal connected set of 1-cells and 0-cells which does not include a vertex constitutes what is called a *side* or *boundary line* (or simply a *boundary*) of the *map* and also of the *regions* on whose complete boundary it may lie. It is clear from this definition that the boundary lines are uniquely determined by the given map. Moreover, a boundary line is evidently a simple arc exclusive of its end points, which are necessarily vertices, or else it is a simple closed curve isolated from all other boundary points (if any) of the map.

If two regions share the same boundary line l as parts of their complete boundaries, they are said to be *contiguous* and to have *contact* with each other across l. If a boundary line l is on the complete boundary of only one region, that region is said to be *self-contiguous* and to have contact with itself across the boundary line l. The occurrence of at least one self-contiguous region is, of course, charactristic of a pseudo-map.

A simply connected non-self-contiguous region having n vertices on its boundary is an *n-gon* or *n-sided region.* For special values of n we have, of course, various synonyms. A 3-gon is a triangle; a 4-gon is a quadrilateral; a 5-gon is a pentagon, and so on.

A complete or partial map is said to be *colored* if to each region of the complete or partial map there is assigned just one color in such a way that no region has the same color as any of the colored regions with which it has contact across boundary lines. The famous four-color conjecture is, in our terminology, to the effect that any complete *proper* map (on a sphere) can be colored in four colors. A *pseudo-map,* of course, can never be colored completely because it always contains at least one region that has contact with itself across a boundary line.

Let P_n be a map of n regions and let $P_n(\lambda)$ denote the number of ways that P_n can be colored using some or all of λ given colors. Then it is well known that $P_n(\lambda)$ can be written as a polynomial in λ, identically zero in the case of a pseudo-map, but otherwise of the nth degree (cf. Birkhoff [2]). These polynomials are called *chromatic polynomials.* In accordance with the notation just used, we shall invariable use the same symbol to indicate a specific map and its associated chromatic polynomial, and the number of regions in the map is *usually* indicated by a subscript. To give an example of the use of this notation we may now state the four-color conjecture in the form: "$K_m(4)\neq 0$, if K_m is any proper map." One of Winn's theorems is to the effect that $K_m(4)\neq 0$, if K_m is any proper map with $m\leqq 35$.

A *constrained* chromatic polynomial means a polynomial which gives the number of ways a certain associated map can be colored under certain restrictions, as for example, that two non-contiguous regions should receive the same color (or perhaps distinct colors), while perhaps certain other regions are not to be colored at all. The regions on which these restrictions are placed will be said to *carry the constraints.* By way of contrast an ordinary chromatic polynomial will be occasionally termed a *free polynomial.* In Chapters V and VI the constrained polynomials furnish a powerful tool for the investigation of the free polynomials, the ultimate object of our study.

Two maps P_n and P_m' are said to be *chromatically equivalent* to each other, if $P_n(\lambda)\equiv P_m'(\lambda)$. Evidently any two pseudo-maps are chromatically equivalent, since in this case both chromatic polynomials are identically zero. In all other cases of chromatic equivalence it is necessary (but not sufficient) that $n=m$, since the degree of the (free) chromatic polynomial is always precisely the same as the number of regions of the map in the case of a proper map.

Two maps are said to be *topologically equivalent* if there exists a homeomorphism which carries the regions and boundaries of one map into the regions and boundaries of the other map. Topological equivalence implies chromatical equivalence, but we shall meet many examples of chromatically equivalent proper maps which are not topologically equivalent.

Two maps are *absolutely* equivalent if the regions and boundaries of one map coincide with the regions and boundaries of the other. Two maps are absolutely, topologically, or chromatically *distinct* if they are not absolutely, topologically, or chromatically *equivalent*.

If two contiguous regions in a map of n regions together with their common boundary are united to form a single region we obtain a modified map, which will have either $n-1$ regions or n regions, according as the two contiguous regions in the original map were distinct or not distinct. The latter case can, of course, happen only if the original map was a pseudo-map. The modified map is called a *submap* of the original map. Moreover the process may be repeated; the submap of a submap is also called a submap of the original map.

This process of forming submaps will be described as the *erasure* or *obliteration* of boundary lines.

It now begins to be clear why it was desirable to make our definition of a map so broad as to include pseudo-maps: namely, the submap of a proper map need not be a proper map. This will always be the case, for instance, if two regions have contact across two or more boundary lines and only one of these boundary lines is erased in forming the submap.

The term *ring* is used in a more general sense than usual, the term *proper ring* being reserved for the sense of the word as hitherto used in the literature on the four color problem. An *n-ring* consists of a closed curve C (which usually need not be specified) without double points, which passes successively through n regions $R_1, R_2, \cdots, R_n$ but which does not pass through any vertices. Here R_i is contiguous with R_{i+1} (i taken modulo n) *but* the n R's need not be distinct and it is not required that R_i should have no contact with R_j ($j \neq i \pm 1$). If, however, the R's are distinct and if there are no contacts between any nonconsecutive two of them (mod n), the ring is called a *proper n-ring*. The *inside* and *outside* of the *ring* refer strictly to the parts of the map inside and outside of C. The n regions $R_1, \cdots, R_n$ are said to *form* the ring.

We shall say that a map is *four-color irreducible*, or, for brevity, *4c. irreducible*, if it can not be colored in four colors, while any proper map with fewer regions can be so colored. In previous papers 4c. irreducibility was simply termed irreducibility. Our purpose in adopting the newer term is that in the future other types of irreducibility will probably play an important role. In view of our conjecture of §2, Chapter IV, one might, for instance, define a map P_{n+3} of triple vertices and simply connected regions to be *absolutely irreducible*, if the relations,

$$(\lambda - 3)^n \ll \frac{P_{n+3}(\lambda)}{\lambda(\lambda - 1)(\lambda - 2)} \ll (\lambda - 2)^n \qquad \text{for } \lambda \geqq 4,$$

do not all hold but corresponding relations do hold for any proper map of simply connected regions and triple vertices with fewer than $n+3$ regions.

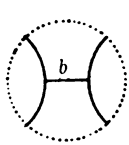
A map which is not irreducible (in a particular sense) is said to be *reducible* (in the same sense). A *configuration* of regions is said to be *reducible* if its presence in a proper map implies that the map is reducible.

The term *reduction formula* first introduced in §2, Chapter I, has little to do with reducibility in the senses discussed above. It is used merely to denote a fairly general formula which may be used to express a chromatic polynomial associated with a map of n regions in terms of polynomials associated with maps having fewer than n regions.

A *scheme* for a set of regions (usually forming a ring) is a rule which divides the regions of the set into subsets of noncontiguous regions and requires that all regions of a subset be colored alike but unlike the regions of any other subset. The constrained polynomials mentioned above usually, though not invariably, occur in connection with schemes. The word scheme in this technical sense does not play an essential role until §2 of Chapter V. It should be mentioned that the word is used in a somewhat different sense in previous work (cf. Birkhoff [1]).

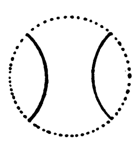
CHAPTER I. FIRST PRINCIPLES IN THE NUMERICAL AND THEORETICAL TREATMENT OF CHROMATIC POLYNOMIALS

1. The three fundamental principles. Throughout this chapter, unless the contrary is clearly indicated, all maps are assumed to have triple vertices only. All results are still true for more general maps, but this convention greatly simplifies the statement of most of our essential results.



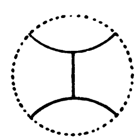
P_n

FIG. 1



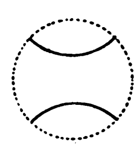
P_{n-1}

FIG. 2



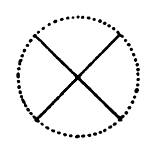
P_n^*

FIG. 3



P_{n-1}^*

FIG. 4



Q_n

FIG. 5

A large part of the theory of chromatic polynomials can be based on three very simple formulas, namely:

$$P_n(\lambda) = (\lambda - 2)P_{n-1}(\lambda), \tag{1.1}$$

$$P_n(\lambda) = (\lambda - 3)P_{n-1}(\lambda), \tag{1.2}$$

$$P_n(\lambda) + P_{n-1}(\lambda) = P_n^*(\lambda) + P_{n-1}^*(\lambda). \tag{1.3}$$

In formula (1.1), P_n is any map having a 2-gon and P_{n-1} is the map obtained by erasing from P_n one of the sides of this 2-gon. In the future, we shall need to use this formula also in the case when one (but not both) of the vertices of the 2-gon has multiplicity greater than 3, in which case the formula is obviously still valid. In formula (1.2), P_n is any map having a 3-gon and P_{n-1} is the map obtained by erasing from P_n one of the sides of this 3-gon. In formula (1.3), the maps P_n, P_{n-1}, P_n^*, P_{n-1}^* are illustrated in figures 1, 2, 3, 4, respectively. Figure 1 represents any map P_n in which we isolate a given boundary b by drawing a dotted simple closed curve completely around it. Each of the other figures, 2 to 5, represents the same map appropriately modified within the dotted curve. The part of the map exterior to the dotted curve is the same for each of the five maps(4).

The proofs of formulas (1.1) and (1.2) are immediately obvious and are therefore omitted.

The truth of formula (1.3) also becomes obvious when we note that any coloration of P_n (fig. 1) or of P_{n-1} (fig. 2) yields a coloration of Q_n (fig. 5); and, conversely, any coloration of Q_n corresponds to a coloration of either P_n or P_{n-1} according as the two colors above and below the quadruple vertex are distinct or the same. Hence $Q_n(\lambda)=P_n(\lambda)+P_{n-1}(\lambda)$. Similarly, $Q_n(\lambda)=P_n^*(\lambda)+P_{n-1}^*(\lambda)$, and formula (1.3) is proved.

It may be remarked that in a certain sense our three fundamental formulas are not independent. Either one of the formulas (1.1) or (1.2) can be deduced from the other with the help of (1.3) without using any other fundamental principle except the axioms of elementary algebra and the convention that the chromatic polynomial of a pseudo-map is zero. For the sake of simplicity it seems wise to give no further attention to this aesthetic defect.

It would be extremely cumbersome to maintain a strictly consistent notation throughout this paper; and, indeed, a definite inconsistency in notation is already to be noted with respect to formulas (1.1), (1.2), and (1.3), where, for example, P_n does not necessarily refer to the same map in the three cases. Hence, we shall think of our three formulas as embodying three simple fundamental principles, rather than as formulas in which we shall, more or less blindly, carry out mechanical substitutions. We shall refer to these principles as *Principles* (1.1), (1.2), and (1.3) respectively.

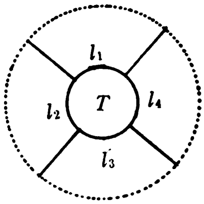
2. **The quadrilateral reduction formula.** From the three fundamental principles discussed above we can deduce a number of very important *reduction formulas* of which the simplest is the following:

$$(2.1) \qquad P_n(\lambda) = (\lambda - 4)P_{n-1}(\lambda) + (\lambda - 3)P_{n-2}(\lambda) + \overline{P}_{n-2}(\lambda),$$

which may be applied to any map P_n (fig. 6) containing a four-sided region T. If the sides of T are l_1, l_2, l_3, l_4, in cyclic order, the map P_{n-1} is formed from P_n by erasing l_1 (or l_3); the map P_{n-2} is formed by erasing both l_1 and l_3; $\overline{P}_{n-2}$ is formed by erasing both l_2 and l_4.

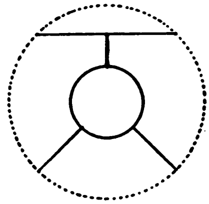
(4) C. N. Reynolds, in an unpublished work, describes the modification of P_n (fig. 1) which yields P_n^* (fig. 3) as a "twisting of the boundary b." The map P_{n-1} (fig. 2) is obtained by erasing b from P_n, and P_{n-1}^* (fig. 4) is obtained by erasing b from P_n^*. It is understood, of course, that some, or even all, of the maps represented in these five figures may be pseudo-maps.

In the proof of this "quadrilateral reduction formula," we need also the map $\overline{P}_{n-1}$ obtained by erasing l_2 (or l_4), as well as the maps P_n^* and P_{n-1}^* indi-



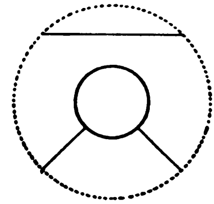
P_n

FIG. 6



P_n^*

FIG. 7



P_{n-1}^*

FIG. 8

cated in figures 7 and 8, whereby it is understood that the part of the map exterior to the dotted curve is the same for each of the three maps P_n (fig. 6), P_n^* (fig. 7), and P_{n-1}^* (fig. 8). By Principle (1.3) we evidently have

$$P_n(\lambda) = P_n^*(\lambda) + P_{n-1}^*(\lambda) - P_{n-1}(\lambda), \tag{2.2}$$

while from Principles (1.1) and (1.2) we have $P_n{}^*(\lambda) = (\lambda - 3)\overline{P}_{n-1}(\lambda)$ and $P_{n-1}^*(\lambda) = (\lambda - 2)\overline{P}_{n-2}(\lambda)$. Hence (2.2) becomes

$$P_n(\lambda) = (\lambda - 3)\overline{P}_{n-1}(\lambda) - P_{n-1}(\lambda) + (\lambda - 2)\overline{P}_{n-2}(\lambda). \tag{2.3}$$

Again using (1.3), we have $\overline{P}_{n-1}(\lambda) = P_{n-1}(\lambda) + P_{n-2}(\lambda) - \overline{P}_{n-2}(\lambda)$. Substituting this value of $\overline{P}_{n-1}(\lambda)$ in (2.3), we obtain the desired formula (2.1).

Similarly, by advancing the subscripts of l_1, l_2, l_3, l_4, we have the formula

$$P_n(\lambda) = (\lambda - 4)\overline{P}_{n-1}(\lambda) + (\lambda - 3)\overline{P}_{n-2}(\lambda) + P_{n-2}(\lambda). \tag{2.4}$$

Hence, adding (2.1) and (2.4) and dividing by 2, we get the more symmetrical formula

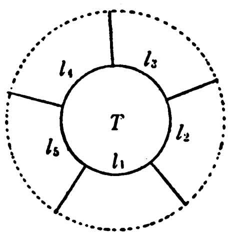
$$P_n(\lambda) = \frac{\lambda - 4}{2}\left[P_{n-1}(\lambda) + \overline{P}_{n-1}(\lambda)\right] + \frac{\lambda - 2}{2}\left[P_{n-2}(\lambda) + \overline{P}_{n-2}(\lambda)\right], \tag{2.5}$$

first proved by another method in 1930 (cf. Birkhoff [3]; also Birkhoff [4, pp. 14, 15]). It is evident that formulas (2.1) and (2.5) are almost equivalent, as either one can be derived from the other by applying Principle (1.3). We shall accordingly call (2.5) the *symmetrical form* of the quadrilateral reduction formula whereas (2.1), or (2.4), will be referred to as the *simplest nonsymmetrical form*. Still another nonsymmetrical form is given by (2.3). This form, however, is not used very much as it has the disadvantage, as compared with (2.1), of involving two maps of $n-1$ regions and only one map of $n-2$ regions, instead of the other way around. The symmetrical form has the disadvantage of involving fractions as well as four maps, instead of only three, on the right-hand side.

3. The pentagon reduction formula. Analogous to the results of the preceding paragraph we have various forms for a pentagon reduction formula. The simplest nonsymmetrical form is

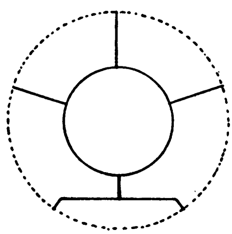
$$P_n(\lambda) = (\lambda - 5)P^1_{n-1}(\lambda) + (\lambda - 4)[P^2_{n-2}(\lambda) + P^5_{n-2}(\lambda)] + P^1_{n-2}(\lambda) + P^3_{n-2}(\lambda) + P^4_{n-2}(\lambda), \tag{3.1}$$

which may be applied to any map P_n (cf. fig. 9) containing a five-sided region T. Denoting the sides of T by l_1, l_2, l_3, l_4, l_5 in cyclic order, P^i_{n-1} refers to the map obtained by erasing l_i, while P^i_{n-2} refers to the map obtained by erasing l_{i-1} and l_{i+1}. Here the indices are taken modulo 5.



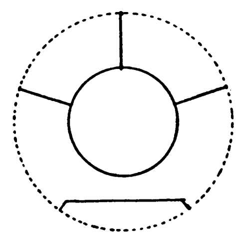
P_n

FIG. 9



P^*

FIG. 10



P^*_{n-1}

FIG. 11

In addition to these maps we need, for the proof of (3.1), the maps P^*_n (fig. 10) and P^*_{n-1} (fig. 11), whereby it is understood that the part of the map exterior to the dotted curve is the same for each of the three maps represented by figures 9, 10, and 11. By Principle (1.3) we evidently have

$$P_n(\lambda) = P^*_n(\lambda) + P^*_{n-1}(\lambda) - P^1_{n-1}(\lambda). \tag{3.2}$$

But the map $P_n{}^*$ (fig. 10) has a four-sided region. Hence the quadrilateral reduction formula (2.1) can be applied and we get

$$P^*_n(\lambda) = (\lambda - 4)P^2_{n-1}(\lambda) + (\lambda - 3)P^3_{n-2}(\lambda) + P^4_{n-2}(\lambda).$$

Also P^*_{n-1} (fig. 11) has a three-sided region, so that (1.2) may be applied and we get $P^*_{n-1}(\lambda) = (\lambda - 3)P^1_{n-2}(\lambda)$. Inserting these values of $P^*_n(\lambda)$ and $P^*_{n-1}(\lambda)$ into (3.2) we have

$$P_n(\lambda) = (\lambda - 4)P^2_{n-1}(\lambda) - P^1_{n-1}(\lambda) + (\lambda - 3)P^1_{n-2}(\lambda) + (\lambda - 3)P^3_{n-2}(\lambda) + P^4_{n-2}(\lambda). \tag{3.3}$$

Now it can easily be shown with the help of Principle (1.3) that $P^1_{n-1}(\lambda) + P^5_{n-2}(\lambda) = P^3_{n-1}(\lambda) + P^4_{n-2}(\lambda)$ together with four other equations obtained by advancing the indices (modulo 5). It follows that $P^1_{n-1}(\lambda) = P^2_{n-1}(\lambda) + P^1_{n-2}(\lambda) - P^2_{n-2}(\lambda) + P^3_{n-2}(\lambda) - P^5_{n-2}(\lambda)$. Inserting this value of $P^1_{n-1}(\lambda)$ into (3.3) we obtain

$$P_n(\lambda) = (\lambda - 5)\overset{2}{P}_{n-1}(\lambda) + (\lambda - 4)\overset{1}{P}_{n-2}(\lambda) + \overset{2}{P}_{n-2}(\lambda) + (\lambda - 4)\overset{3}{P}_{n-2}(\lambda) + \overset{4}{P}_{n-2}(\lambda) + \overset{5}{P}_{n-2}(\lambda),$$

the required formula (3.1) with the indices advanced one unit (modulo 5).

The symmetrical form of the pentagon reduction formula is obtained by adding (3.1) to the four other equalities obtained by advancing the indices (modulo 5). After dividing by 5, the result is

$$(3.4)\qquad P_n(\lambda) = \frac{\lambda - 5}{5}\left[\sum_{i=1}^{5} \overset{i}{P}_{n-1}(\lambda)\right] + \frac{2\lambda - 5}{5}\left[\sum_{i=1}^{5} \overset{i}{P}_{n-2}(\lambda)\right],$$

a formula first proved by another method in 1930 (cf. Birkhoff [3, 4], loc. cit.).

(3.1), (3.3) and (3.4) are equivalent in the sense that any two of them can be derived from the third by use of Principle (1.3). Although (3.3) involves only five maps on the right-hand side, two of these have $n-1$ regions. This fact and the occurrence of a strictly negative term for $\lambda \geqq 5$ in (3.3) combine to make (3.3) less useful than (3.1). Hence the expression "*simplest*" nonsymmetrical form is used with reference to the latter formula.

4. **The m-gon reduction formula.** The methods used in deducing the quadrilateral and pentagon reduction formulas will evidently yield an m-gon reduction formula. Although the indicated proof of a general formula by complete induction appears to be very complicated and has not yet been explicitly constructed, the preceding statement is certainly true with regard to any preassigned numerical value of m. The following is an entirely different method of proof, somewhat reminiscent of the original method used by Birkhoff in 1930 (loc. cit.) for the special cases $m=4$ and 5. This proof rests on ideas somewhat more complicated than the three principles of §1.

The symmetrical and nonsymmetrical forms of the m-gon formula are given respectively by Theorems I and II below:

THEOREM I. *Let T be an m-gon in a map P_n of n regions. Let $\Pi_{n-k}(\lambda)$ denote the sum of the chromatic polynomials associated with the submaps obtained by erasing just k boundary lines of T. Then*

$$(4.1)\qquad P_n(\lambda) = \frac{1}{m}\sum_{k=1}^{[m/2]} (k\lambda - m)\Pi_{n-k}(\lambda),$$

where $[m/2]=m/2$ or $m/2-1/2$ according as m is even or odd.

THEOREM II. *Let the boundaries of T be denoted by $l_1, l_2, \cdots, l_m$ in cyclic order. Let $\overset{i}{U}_{n-k}(\lambda)$ denote the sum of the chromatic polynomials associated with the submaps obtained by erasing the boundary l_i together with just $k-1$ other boundaries of T. Then, with the notation of the previous theorem,*

$$(4.2)\qquad P_n(\lambda) = (\lambda - m)\sum_{k=1}^{[m/2]} \overset{i}{U}_{n-k}(\lambda) + \sum_{k=2}^{[m/2]} (k-1)\Pi_{n-k}(\lambda),$$

for any positive integral value of i from 1 *to m, inclusive.*

It may be remarked that not only are formulas (2.1), (2.5), (3.1), and (3.4) special cases of (4.1) and (4.2) but so also are (1.1) and (1.2), as we would see by taking $m=2$ and 3 respectively.

Proof of Theorem I. The symbol $p[\alpha_1, \beta_1, \alpha_2, \beta_2, \cdots, \alpha_s, \beta_s]$, in which the α's and β's are positive integers with $\alpha_1 > \alpha_2 > \cdots > \alpha_s > 0$, is of central importance. Its value is defined as follows: Let $p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s]$ equal the number of ways $P_n - T$ can be colored (in λ colors) so that β_i distinct colors each occur just α_i times in the ring of regions surrounding T. From this definition it follows that

(4.3) $\beta_1 + \beta_2 + \cdots + \beta_s =$ total number of colors occurring in the ring in each of the colorations here enumerated;

$$\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_s\beta_s = m, \tag{4.4}$$

at least if $p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s] \neq 0$. This shows, in particular, that s is bounded, say not greater than m. Also α and β are obviously bounded.

It is easy to see from (4.3) that

$$P_n(\lambda) = \sum_s \sum_{\alpha,\beta} (\lambda - \beta_1 - \beta_2 - \cdots - \beta_s) p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s], \tag{4.5}$$

where $\sum_{\alpha,\beta}$ denotes a sum, for fixed s, over all possible sets of values for $\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s$. $\sum_s$ represents a sum over all possible values of s. Also we see that

$$\Pi_{n-k}(\lambda) = \sum_s \sum_{i=1}^{s} \sum_{\alpha_i = k} \beta_i p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s], \tag{4.6}$$

where $\sum_{\alpha_i=k}$ denotes a sum, for fixed s and i, over all possible values of $\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s$ such that $\alpha_i = k$. Multiplying (4.6) by $k\lambda - m$ and summing over all possible values of k, we have

$$\begin{aligned}
\sum_k (k\lambda - m)\Pi_{n-k}(\lambda) &= \sum_k (k\lambda - m) \sum_s \sum_{i=1}^{s} \sum_{\alpha_i=k} \beta_i p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s] \\
&= \sum_k \sum_s \sum_{i=1}^{s} \sum_{\alpha_i=k} (\alpha_i\lambda - m)\beta_i p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s] \\
&= \sum_s \sum_{i=1}^{s} \sum_k \sum_{\alpha_i=k} (\alpha_i\beta_i\lambda - m\beta_i) p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s] \\
&= \sum_s \sum_{i=1}^{s} \sum_{\alpha,\beta} (\alpha_i\beta_i\lambda - m\beta_i) p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s] \\
&= \sum_s \sum_{\alpha,\beta} \sum_{i=1}^{s} (\alpha_i\beta_i\lambda - m\beta_i) p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s],
\end{aligned}$$

and this, by (4.4), yields

$$\sum_{k=1}^{[m/2]} (k\lambda - m)\Pi_{n-k}(\lambda) = \sum_{s}\sum_{\alpha,\beta} (m\lambda - m\beta_1 - m\beta_2 - \cdots - m\beta_s)p[\alpha_1, \beta_1, \cdots, \alpha_s, \beta_s]. \tag{4.7}$$

The upper limit for the summation over k is $[m/2]$, since the erasure of more than half of the boundaries $l_1, l_2, \cdots, l_m$ of T could not possibly yield a proper submap of P_n. In fact, the erasure of more than half the boundaries of T would involve the erasure of at least two consecutive boundaries and this would leave a free vertex.

Comparison of (4.7) with (4.5) yields the result that

$$mP_n(\lambda) = \sum_{k=1}^{[m/2]} (k\lambda - m)\Pi_{n-k}(\lambda).$$

Division of this last equation by m completes the proof of Theorem I.

Proof of Theorem II. We introduce the map Ω_{n-1} obtained from the original map P_n by shrinking T to a point in such wise that the m regions in the ring about T abut on a single m-tuple vertex. Let $\Omega_{n-1}(\lambda)$ denote the corresponding chromatic polynomial. Then evidently

$$\Omega_{n-1}(\lambda) = \sum_{k=1}^{[k/2]} U^i_{n-k}(\lambda) \tag{4.8}$$

independently of i, inasmuch as $U^i_{n-k}(\lambda)$ is equal to the number of ways of coloring Ω_{n-1} in such a manner that the region R_i, which originally had contact with T across the boundary l_i, receives the same color as exactly $k-1$ of the other regions about the m-tuple vertex. It is also clear from the definition of $U^i_{n-k}(\lambda)$ that

$$\sum_{i=1}^{m} U^i_{n-k}(\lambda) = k\Pi_{n-k}(\lambda), \tag{4.9}$$

inasmuch as each submap obtained by erasing just k boundaries of T will be represented just k times in the sum on the left.

Taking $i=j$ in (4.8) and subtracting the result from (4.8) in its original form, we obtain an equality which we solve for $U^i_{n-1}(\lambda)$ in terms of the other U's,

$$U^i_{n-1}(\lambda) = U^j_{n-1}(\lambda) + \sum_{k=2}^{[m/2]} [U^j_{n-k}(\lambda) - U^i_{n-k}(\lambda)]. \tag{4.10}$$

By taking $k=1$, we obtain from (4.9)

$$\Pi_{n-1}(\lambda) = \sum_{i=1}^{m} U^i_{n-1}(\lambda).$$

Hence, substituting from (4.10), we get

$$\Pi_{n-1}(\lambda) = mU^{j}_{n-1}(\lambda) + \sum_{i=1}^{m} \sum_{k=2}^{[m/2]} [U^{j}_{n-k}(\lambda) - U^{i}_{n-k}(\lambda)].$$

Hence, reversing the order of summation and making use of (4.9), we have

$$\Pi_{n-1}(\lambda) = m \sum_{k=1}^{[m/2]} U^{j}_{n-k}(\lambda) - \sum_{k=2}^{[m/2]} k\Pi_{n-k}(\lambda).$$

Inserting this value of $\Pi_{n-1}(\lambda)$ into (4.1), we get

$$\begin{aligned} P_n(\lambda) &= \frac{1}{m}\left\{(\lambda - m)\left[m\sum_{k=1}^{[m/2]} U^{j}_{n-k}(\lambda) - \sum_{k=2}^{[m/2]} k\Pi_{n-k}(\lambda)\right] + \sum_{k=2}^{[m/2]} (k\lambda - m)\Pi_{n-k}(\lambda)\right\} \\ &= (\lambda - m)\sum_{k=1}^{[m/2]} U^{j}_{n-k}(\lambda) + \sum_{k=2}^{[m/2]} \frac{km - m}{m}\Pi_{n-k}(\lambda). \end{aligned}$$

Hence $P_n(\lambda) = (\lambda - m)\sum_{k=1}^{[m/2]} U^{j}_{n-k} + \sum_{k=2}^{[m/2]}(k-1)\Pi_{n-k}(\lambda)$. But this is exactly the formula (4.2) which we desired to prove with i replaced by j.

5. **On the number of terms in the sums represented by Π_{n-k} and U^{i}_{n-k}, which occur in the m-gon formula.** We wish to find the number $F(m, k)$ of absolutely distinct proper submaps that can be formed by erasing just k of the m boundaries of the region T in the map P_n. For this purpose we assume that the ring surrounding T is a proper ring. $F(m, k)$ is then evidently independent of the part of P_n exterior to the ring. It is, in fact, a function of the integers m and k alone. In order to visualize the problem more clearly, we note that $F(m, k)$ = *the number of ways in which any proper ring of m regions can be colored in black and white in such wise that no two black regions shall be in contact* (but without regard to whether or not two white regions are in contact) *and so that just k regions are colored black.*

Let us first consider three consecutive regions A, B, C in a ring R_{m+1} of $m+1$ regions. The colorations of R_{m+1} ennumerated by $F(m+1, k)$ fall into just five types according to the way in which A, B, C are colored. These types are indicated in the following table:

type	A	B	C	number of other black regions in R_{m+1}
1	white	white	white	k
2	black	white	white	$k-1$
3	white	white	black	$k-1$
4	white	black	white	$k-1$
5	black	white	black	$k-2$

If we shrink the region B until it disappears in such a way that A and C abut each other, we get a ring R_m of m regions. If in R_m we merge the abutting regions A and C into a single region, we get a ring R_{m-1} of $m-1$ regions. It is evident from the above table that every coloration of R_{m+1} in types 1, 2, or 3 yields just one coloration of R_m with k black regions, and conversely. Likewise every coloration in types 4 and 5 yields one coloration of R_{m-1} with $k-1$ black regions, and conversely. It follows that

$$F(m + 1, k) = F(m, k) + F(m - 1, k - 1). \tag{5.1}$$

The argument which led to this partial difference equation is valid as long as $m \geqq 2$ and $k \geqq 0$, although the number of colorations in types 2, 3, 4, 5 would be zero for $k=0$ and the number in type 5 would also be zero for $k=1$. Actually we only need the validity of (5.1) for $k \geqq 1$. We also obviously have

$$F(m, 0) = 1, \qquad m \geqq 1, \tag{5.2}$$

$$F(2k, k) = 2, \qquad k \geqq 1. \tag{5.3}$$

It is easy to see that the three equations (5.1), (5.2), and (5.3) determine our function $F(m, k)$ uniquely for integral values of m and k satisfying $m \geqq 2k > 0$. Hence, we can verify a posteriori that

$$F(m, k) = C_k^{m-k+1} - C_{k-2}^{m-k-1} \tag{5.4}$$

where C_q^p is the coefficient of x^q in the expansion of $(1+x)^p$. For, in virtue of the known properties of the binomial coefficients, particularly the relation $C_q^{p+1} = C_q^p + C_{q-1}^p$, it is easy to show that (5.4) satisfies (5.1), (5.2), and (5.3).

We are also interested in finding the number $G(m, k)$ of absolutely distinct proper submaps to be formed by erasing a preassigned boundary of T together with just $k-1$ other boundaries, assuming again that the ring surrounding T is a proper ring. As before, we obtain

$$G(m + 1, k) = G(m, k) + G(m - 1, k - 1),$$

$$G(m, 1) = 1, \qquad m \geqq 1,$$

$$G(2k, k) = 1, \qquad k \geqq 1.$$

It follows from these three equations that

$$G(m, k) = C_{k-1}^{m-k-1} = (k/m)\, F(m, k). \tag{5.5}$$

Thus, in the general case in which the ring surrounding T is a proper ring, the *expressions* $\Pi_{n-k}(\lambda)$ *and* $U_{n-k}^1(\lambda)$ *which occur in* (4.1) *and* (4.2) *represent sums of* $F(m, k)$ *and* $G(m, k)$ *chromatic polynomials respectively, where* $F(m, k)$ *and* $G(m, k)$ *may be expressed in terms of binomial coefficients as indicated by* (5.4) *and* (5.5).

6. **A general reduction theorem.** We now consider a general reduction theorem of which Theorem II of §4 is an explicit example.

THEOREM I. *Let R_ν denote a ring (not necessarily proper) of ν regions, of which μ are distinct ($\mu \leqq \nu$) in a map P_n which has α regions on one side (say exterior) of the ring and β regions on the other side (interior), so that $\alpha+\beta+\mu=n$, the total number of regions in P_n. Let P^1, P^2, $\cdots$ denote the various submaps, of not more than $\alpha+\mu$ regions each, that can be obtained by erasing boundaries wholly interior to R_ν. Then the chromatic polynomial $P_n(\lambda)$ associated with P_n can be expressed in the form*

$$P_n(\lambda) = \sum_i A^i(\lambda)P^i(\lambda), \tag{6.1}$$

where $A^1(\lambda)$, $A^2(\lambda)$, $\cdots$ are polynomials in λ with integral coefficients, which are entirely independent of the configuration exterior to the ring R_ν.

Proof. We prove the theorem by induction on β. The theorem is trivial if $\beta=0$, because then $P_n(\lambda)$ can be regarded as a submap of itself of not more than $\alpha+\mu$ regions and hence we may take $P^1(\lambda)=P_n(\lambda)$, $A^1(\lambda)\equiv 1$, $A^2(\lambda) \equiv A^3(\lambda) \equiv \cdots \equiv 0$. We therefore assume inductively that the theorem is true when $\beta \leqq \gamma-1$ $(\gamma \geqq 1)$. We shall prove that the theorem must then be true when $\beta=\gamma$.

Since P_n has $\beta=\gamma>0$ regions interior to the ring R_ν, there must exist at least one region T lying wholly within the ring. The proper submaps Q^1, Q^2, $\cdots$ obtained by erasing the various boundaries of T have the same configuration exterior to R_ν that P_n has, but, interior to R_ν, each has at most $\gamma-1$ regions. Hence, by our inductive hypothesis and the fact that the submaps of Q^i must also be submaps of P_n, we have

$$Q^j(\lambda) = \sum_i A^{ij}P^i(\lambda), \qquad j = 1, 2, \cdots, \tag{6.2}$$

where $A^{ij}(\lambda)$ is a polynomial in λ with integral coefficients independent of the exterior of R_ν. On the other hand, we know from Theorem II of §4 that we can always write an identity of the form

$$P_n(\lambda) = \sum_j L^j(\lambda)Q^j(\lambda), \tag{6.3}$$

where $L^1(\lambda)$, $L^2(\lambda)$, $\cdots$ are polynomials in λ with integral coefficients (of degree not greater than 1) depending only on the number, m, of boundaries of T, and hence wholly independent of the exterior of R_ν. If, in (6.3), we substitute for $Q^j(\lambda)$ its value given by (6.2), we arrive at an identity of the form (6.1) with $A^i(\lambda)=\sum_j A^{ij}(\lambda)L^j(\lambda)$. Since both A^{ij} and L^j have integral coefficients and are independent of the exterior of R_ν, the same is true of $A^i(\lambda)$ and our proof by induction is complete.

It is important to notice for future application that this *theorem is also valid for constrained chromatic polynomials provided only that the constraints are not carried by any regions completely interior to R_ν. In such cases, the coefficients $A^i(\lambda)$ are independent of the nature of the constraints.*

Evidently, as already noted, Theorem II of §4 is an explicit example of the general theorem just proved with $\nu=m$ and $\beta=1$. We shall pause to give another nontrivial example with $\nu=6$ and $\beta=4$, the interior of the ring R_6 consisting of four pentagons (cf. fig. 12). By erasing boundaries inside the

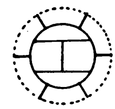


FIG. 12

ring we get the submaps topologically equivalent to those indicated in figure 13. The chromatic polynomial $P_n(\lambda)$ associated with the map of figure 12

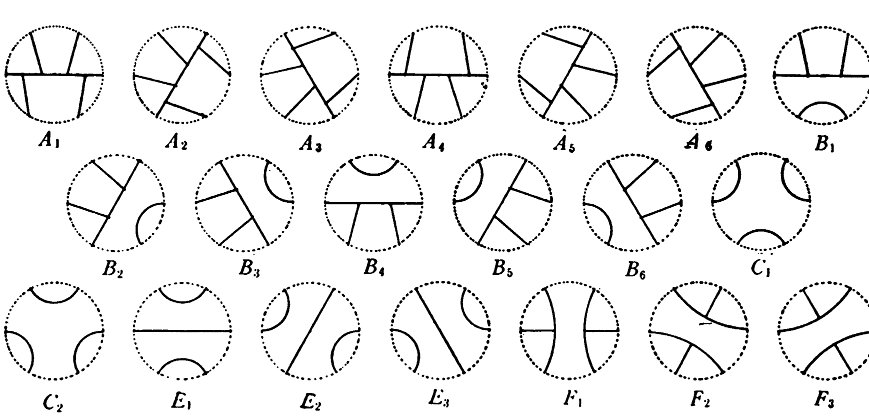


FIG. 13

is then given in terms of the chromatic polynomials of the submaps by means of the formula

$$
\begin{aligned}
P_n(\lambda) = {} & (u^4 - 3u^3 + 10u^2 - 16u + 8)A_1(\lambda) + (u^3 - 4u^2 + 9u - 4)B_1(\lambda) \\
& + (u^4 - 3u^3 + 9u^2 - 12u + 5)B_2(\lambda) + (-u^2 + 4u - 3)B_3(\lambda) \\
& + (u^3 - 4u^2 + 9u - 4)B_4(\lambda) + (-u^2 + 4u - 3)B_5(\lambda) \\
& + (u^4 - 3u^3 + 9u^2 - 12u + 5)B_6(\lambda) \\
& + (u^3 - 6u^2 + 11u - 6)C_1(\lambda) + (u^4 - 2u^3 + 4u^2 - 5u + 2)C_2(\lambda) \\
& + (u^2 - 5u + 2)E_1(\lambda) + E_2(\lambda) + E_3(\lambda) \\
& + (u^4 - 3u^3 + 10u^2 - 14u + 6)F_1(\lambda) + (-3u^2 + 6u - 3)F_2(\lambda) \\
& + (-3u^2 + 6u - 3)F_3(\lambda),
\end{aligned} \tag{6.4}
$$

where $u=\lambda-3$. The deduction of this formula along the lines indicated in the proof of Theorem I, or by direct use of the three fundamental principles of

§1, is routine but extremely laborious. Hence the actual calculations are omitted. One reason for taking the trouble of calculating such a formidable formula in the first place was the following:

It is known from a study of the Kempe chains (cf. Birkhoff [1, p. 126]) that the configuration of four pentagons surrounding a boundary is a 4c. reducible configuration. The question arose as to whether or not the irreducibility of this, and similar configurations, could be proved also by means of a reduction formula of the type discussed in Theorem I. The answer is in the negative. For, if we put $\lambda=4$ (that is, $u=1$) in (6.4) we get

$$P_n(4) = 2B_1(4) + 2B_4(4) - 2E_1(4) + E_2(4) + E_3(4), \tag{6.5}$$

and the negative term $-2E_1(4)$ prevents us from concluding that $P_n(4)$ is positive.

Formula (6.5) is, however, not entirely useless; it enables us to prove the following result, of some interest, perhaps, but of very minor importance:

THEOREM II. *If the map* B_1 (fig. 13) *is colorable in four colors, then*

$$2B_1(4) + 2B_4(4) + E_2(4) + E_3(4) > 2E_1(4). \tag{6.6}$$

The proof consists in the remark that Birkhoff's result (loc. cit.), although not explicitly stated in this way, is to the effect that, if B_1 is colorable in four colors, then P_n is also colorable in four colors, that is, $P_n(4)>0$. Our inequality (6.6) will then follow at once from (6.5).

In spite of the disappointingly weak results obtainable by formulas of the type indicated by Theorem I, it will be seen later in Chapter V that the methods of the present chapter are indeed available for proving many, if not all, of the results previously proved only by use of the Kempe chain theory. This is done by considering not only the free chromatic polynomials but the constrained chromatic polynomials as well.

7. **The reduction of the 2- and 3-rings.** We close this chapter by recording two obvious formulas (cf. Birkhoff [4]) for the reduction of the 2-ring and 3-ring. They are

$$P_n(\lambda) = \frac{P_\alpha(\lambda)\cdot P_\beta(\lambda)}{\lambda(\lambda-1)}, \qquad \alpha+\beta-2=n, \tag{7.1}$$

for the 2-ring, and

$$P_n(\lambda) = \frac{P_\alpha(\lambda)\cdot P_\beta(\lambda)}{\lambda(\lambda-1)(\lambda-2)}, \qquad \alpha+\beta-3=n, \tag{7.2}$$

for the 3-ring. In either case P_α is the map obtained by shrinking to a point the part of the map P_n which lies on one side of the ring, while P_β is formed likewise by shrinking to a point the other side of the map. These formulas may be regarded as generalizations of (1.1) and (1.2).

These reduction formulas are very simple because all the regions forming a 2-ring or a 3-ring are in mutual contact with each other and hence their colors must be mutually distinct. Therefore, to match any coloration of P_α with a given coloration of P_β it is only necessary to permute the colors of P_α. This is, of course, not true for the m-ring ($m \geqq 4$). Hence the analogue of (7.1) and (7.2) for even the 4-ring is much more complicated. We have succeeded (cf. chapters V and VI) in obtaining these analogues for both the 4-ring and the 5-ring, but not for the m-ring—neither for the general case nor for any explicit value of $m \geqq 6$. If we were in possession of the completely general formula, the four-color problem could probably be considered substantially on the way toward solution.

Chapter II. The explicit computation of chromatic polynomials

1. Preliminary remarks and explanation of the method of computation. It seemed desirable to have on hand a fairly large assortment of chromatic polynomials in the hope that they might give some insight into the general theory. Unfortunately, except for a few special cases, the explicit computation of these polynomials is very involved for maps with a fairly large number of regions. The computation may be somewhat simplified by introducing the following.

$$u = \lambda - 3,$$

$$Q_n(u) = \frac{P_n(\lambda)}{\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)}.$$

Now $P_n(0) = P_n(1) = P_n(2) = 0$ for any map containing at least one (triple) vertex and $P_n(3) = 6$, or 0, according as the map (all of whose vertices are assumed to be triple) does, or does not, consist entirely of even-sided regions (Heawood [1]). Ordinarily, then, when the map contains at least one odd-sided region, $P_n(\lambda)$ is divisible by $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$ and hence $Q_n(u)$ is a polynomial in u of degree $n-4$. In the relatively few cases when the map contains no odd-sided region, $Q_n(u)$ is a polynomial in u of degree $n-4$ *plus* a term u^{-1}. In either case, we shall call $Q_n(u)$ the *Q polynomial* of the map P_n ($n \geqq 4$).

If we transform the fundamental formulas (1.1), (1.2), (1.3), (2.1), (3.1), (7.1), and (7.2) of Chapter I so that they are expressed in terms of u and the Q polynomials, we get

$$Q_n(u) = (u+1)Q_{n-1}(u), \tag{1.1}$$

$$Q_n(u) = uQ_{n-1}(u), \tag{1.2}$$

$$Q_n(u) + Q_{n-1}(u) = Q_n^*(u) + Q_{n-1}^*(u), \tag{1.3}$$

$$Q_n(u) = (u-1)Q_{n-1}(u) + uQ_{n-2}(u) + Q_{n-2}^*(u), \tag{1.4}$$

$$Q_n(u) = (u-2)\overset{1}{Q}_{n-1}(u) + (u-1)[\overset{2}{Q}_{n-2}(u) + \overset{5}{Q}_{n-2}(u)] + \overset{1}{Q}_{n-2}(u) + \overset{3}{Q}_{n-2}(u) + \overset{4}{Q}_{n-2}(u), \tag{1.5}$$

$$Q_n(u) = u(u+1)Q_\alpha(u)Q_\beta(u), \qquad \alpha+\beta-2=n, \tag{1.6}$$

$$Q_n(u) = uQ_\alpha(u)Q_\beta(u), \qquad \alpha+\beta-3=n. \tag{1.7}$$

The interpretation of these formulas is obvious from §§1, 2, 3, 7 of the previous chapter.

In virtue of (1.1), (1.2), (1.6), and (1.7), the Q polynomial of any map containing a proper 2-ring or 3-ring is easily expressed in factored form in terms of one or two Q polynomials of maps with fewer regions. Hence the only maps whose Q polynomials we list in the table of the next section are those maps (having only triple vertices) which contain no proper 2-rings and no proper 3-rings. Maps of this type are called *regular* maps.

For purposes of classification and reference, it has been found convenient to associate with each regular map a symbol $(n; a, b, c, \cdots)$, where

n = the total number of regions in the map,
a = the number of four-sided regions in the map,
b = the number of five-sided regions in the map,
c = the number of six-sided regions in the map,

and so on. It has been our experience that (for $n \leqq 16$, at least) there are very few topologically distinct maps associated with the same symbol; in many cases, only one. In the cases where there are more than one map associated with the same symbol, the two (or more) maps are easily distinguishable by some easily described topological property. For example, we have two maps associated with the symbol (16; 0, 12, 4), but they are easily distinguished from each other as follows: One of them has four isolated hexagons. The other has two pairs of hexagons, each hexagon of a pair in contact with the other hexagon of the same pair; but the two pairs are isolated from each other.

According to Brückner(5), our list of regular maps is complete up to and including $n=10$. We know, therefore, that our symbol and accompanying topological description characterize our maps uniquely for $n \leqq 10$. Hence it is not necessary to provide further information for these maps.

For $n > 10$, we can not tell (without going through an enormous amount of work) whether or not the map in any given case is uniquely described. Hence it seems necessary to provide a "contact" symbol for the unique characterization of each map listed with more than 10 regions. A contact symbol for a regular map of n regions is a set of $3n-6$ pairs of letters enclosed in brackets. If the symbol contains the pair xy, it means that region x is in contact with region y; otherwise region x is not in contact with region y. With the

(5) Max Brückner, *Vielecke und Vielflache*, Teubner, Leipzig, 1900. Cf. also Reynolds [3]. There are a total of 21 maps given by Brückner as having no 2-sided or 3-sided regions. Three of these have proper 3-rings; eighteen are listed in our table.

help of this symbol it will be possible for the reader to reconstruct a figure representing each map.

In order to facilitate the work of reconstruction, the letters paired with the particular letter a will be given in the same cyclic order in which the corresponding regions surround the region a. Thus the first seven pairs in the contact symbol [ab, ac, ad, ae, af, ag, ah, bc, bj, bi, bh, cd, cj, de, dj, ef, ek, ej, fg, fi, fk, gh, gi, hi, ij, ik, jk] indicate that the region a is surrounded by the ring consisting of regions b, c, d, e, f, g, h, in exactly this cyclic order, so that the further presence of the pairs bc, cd, de, ef, fg, gh, hb is to a certain extent redundant.

The number of sides possessed by a given region is, of course, equal to the number of times the corresponding letter occurs in the symbol. Thus the regions b, c, d, e, f, g, h (in the example just given) are easily seen to have 5, 4, 4, 5, 5, 4, 4 sides respectively. In this particular case this is all the information the reader will need to know in order to reconstruct the map, although the symbol contains much confirmatory further information.

In a systematic calculation of a table, which is to be reasonably complete up to a certain value of n, it is desirable to make the table as complete as possible for $n \leqq p$ before starting with $n = p+1$. Otherwise it is necessary to keep going back to fill in gaps for smaller values of n.

A good illustration of our method will be presented, if we give a complete computation for the chromatic polynomial of the dodekahedron, (12; 0, 12), assuming that the chromatic polynomials for all regular maps of 10 or fewer regions are already known. As a matter of fact it is not necessary to carry the calculation through all of these (of which there are 18) but only a comparatively few (about 7).

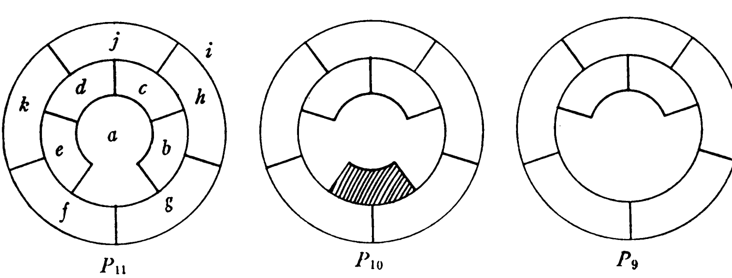


FIG. 14

Let $Q_{12}(u)$ be the chromatic polynomial for the map (12; 0, 12), divided by $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$. Then applying the formula (1.5) to one of the five-sided regions of the dodekahedron, we find that

$$Q_{12}(u) = (u-2)Q_{11}(u) + (2u+1)Q_{10}(u), \tag{1.8}$$

where Q_{11} refers to the map P_{11} of figure 14 and Q_{10} refers to the map P_{10} of figure 14. Now P_{10} has a three-sided region (shaded in the figure). Hence, this map is not listed in our table. We can, however, apply formula (1.2). This gives

$$Q_{10}(u) = uQ_9(u), \tag{1.9}$$

where Q_9 refers to the map P_9 of the figure. P_9 has the symbol (9; 4, 4, 1) and hence we find from our table that $Q_9=u^5+0u^4+5u^3-2u^2+u+1$. Substituting this value for Q_9 in (1.9) we find

$$Q_{10}(u) = u^6 + 0u^5 + 5u^4 - 2u^3 + u^2 + u. \tag{1.10}$$

We next go back to P_{11}. This map has the symbol (11; 2, 8, 1). Moreover, if suitable letters are assigned to its regions (as shown, for example, in figure 14), its contact symbol is readily seen to be the same as that given in the table in connection with (11; 2, 8, 1). Hence our required polynomial, $Q_{11}(u)$, is also found in our table in connection with (11; 2, 8, 1). We are not, however, for

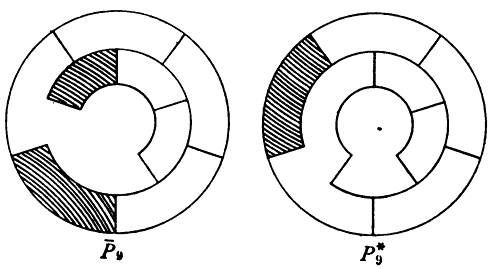


FIG. 15

the purposes of our present example supposing that Q_{11} is known. Instead we apply (1.4) to one of the four-sided regions of P_{11}. We thus find that

$$Q_{11}(u) = (u-1)\overline{Q}_{10}(u) + u\overline{Q}_9(u) + Q_9^*(u), \tag{1.11}$$

where $\overline{Q}_{10}$ refers to the map (10; 3, 6, 1) and hence by our table is given by

$$\overline{Q}_{10}(u) = u^6 + 0u^5 + 6u^4 - 7u^3 + 6u^2 + u - 1, \tag{1.12}$$

while $\overline{Q}_9$ and Q_9^* refer to the maps $\overline{P}_9$ and P_9^* of figure 15. Neither of these two maps is regular. They contain the shaded three-sided regions, which can be reduced by formula (1.2). We thus find that $\overline{Q}_9(u)=u^2Q_7(u)$ and $Q_9^*=uQ_8(u)$, where Q_7 refers to the map (7; 5, 2) and hence is equal to u^3+0u^2+3u+1, while Q_8 refers to the map (8; 4, 4) and hence is equal to $u^4+0u^3+4u^2-u-1$. It follows that

$$\overline{Q}_9(u) = u^5 + 0u^4 + 3u^3 + u^2, \quad Q_9(u) = u^5 + 0u^4 + 4u^3 - u^2 - u. \tag{1.13}$$

Substituting the values of $\overline{Q}_{10}$, $\overline{Q}_9$, Q_9, as given by (1.12) and (1.13), in (1.11), we find that

(1.14) $Q_{11}(u) = u^7 + 0u^6 + 7u^5 - 10u^4 + 18u^3 - 6u^2 - 3u + 1.$

Finally substituting the values of Q_{10} and Q_{11} as given by (1.10) and (1.14) in the equation (1.8), we find that

(1.15) $Q_{12}(u) = u^8 + 0u^7 + 8u^6 - 14u^5 + 39u^4 - 42u^3 + 12u^2 + 8u - 2.$

Anybody wishing to verify this result from the ground up would have to confirm the chromatic polynomials for the following regular maps: (7; 5, 2), (8; 4, 4), (9; 4, 4, 1), (10; 3, 6, 1). Using the same method as that explained above, he would not have a difficult task. Also, it may be noted that (1.15) can be checked in various ways. The dodekahedron possesses a proper 6-ring of such a type that formula (6.4) of Chapter I can be used. Other formulas of this type could also be used for checking purposes. So can formula (1.3).

The polynomial (1.15) was first given by Whitney written in powers of λ (cf. Whitney [2, p. 718]). Whitney's method of computation was not published, though it was made available to the present authors. It was entirely different from the method explained above. So far as is known, this is the only previous case in which a nontrivial chromatic polynomial has been explicitly calculated, certainly the only case published. One reason for this is that most of the coefficients of the powers of λ in the chromatic polynomials are numerically very large for nontrivial maps. This is apt to cause an excessive amount of numerical calculation. This difficulty has been avoided in the present work by writing the polynomials in powers of $\lambda-3$ $(=u)$.

2. Table of chromatic polynomials (divided by $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$) for regular maps.

(6; 6)	$Q_6 = u^2+0u+2+u^{-1}.$
(7; 5, 2)	$u^3+0u^2+3u+1.$
(8; 4, 4)	$u^4+0u^3+4u^2-u-1.$
(8; 6, 0, 2)	$u^4+0u^3+4u^2+3u+3+u^{-1}.$
(9; 3, 6)	$u^5+0u^4+5u^3-5u^2+0u+1.$
(9; 4, 4, 1)	$u^5+0u^4+5u^3-2u^2+u+1.$
(9; 5, 2, 2)	$u^5+0u^4+5u^3+0u^2+2u+1.$
(9; 7, 0, 0, 2)	$u^5+0u^4+5u^3+6u^2+7u+2.$
(10; 2, 8)	$u^6+0u^5+6u^4-8u^3+8u^2+2u-1.$
(10; 3, 6, 1)	$u^6+0u^5+6u^4-7u^3+6u^2+u-1.$
(10; 4, 4, 2)	sixes(6) separate, fives in two mutually separated pairs, $u^6+0u^5+6u^4-2u^3+6u^2+3u+0.$
(10; 4, 4, 2)	sixes separate, fives connected, $u^6+0u^5+6u^4-5u^3+4u^2+0u-1.$
(10; 4, 4, 2)	sixes connected, fives connected, $u^6+0u^5+6u^4-3u^3+6u^2+3u+0.$

(6) Here and in the sequel the words "four," "five," "six," and so on, are used as abbreviations for four-sided, five-sided, six-sided, and so on, regions.

(10; 5, 2, 3)	$u^6+0u^5+6u^4-2u^3+6u^2+3u+0.$
(10; 5, 3, 1, 1)	$u^6+0u^5+6u^4-u^3+3u^2-u-1.$
(10; 6, 0, 4)	$u^6+0u^5+6u^4+u^3+8u^2+7u+4+u^{-1}.$
(10; 6, 2, 0, 2)	$u^6+0u^5+6u^4+2u^3+4u^2-u-1.$
(10; 8, 0, 0, 0, 2)	$u^6+0u^5+6u^4+10u^3+14u^2+8u+4+u^{-1}.$
(11; 2, 8, 1)	[*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, cj, ch, de, dk, dj, ef, ek, fg, fi, fk, gh, gi, hi, hj, ki, kj, ij*] $u^7+0u^6+7u^5-10u^4+18u^3-6u^2-3u+1.$
(11; 3, 6, 2)	fours isolated [*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, ck, ch, de, dk, ef, ej, ek, fg, fi, fj, gh, gi, hk, hj, hi, kj, ij*] $u^7+0u^6+7u^5-9u^4+14u^3-5u^2-2u+1.$
(11; 3, 6, 2)	pair of fours in contact [*ab, ac, ad, ae, af, ag, bc, bi, bh, bg, cd, ck, ci, de, dk, ef, ek, fg, fj, fk, gh, gj, hi, hj, ik, ij, jk*] $u^7+0u^6+7u^5-8u^4+13u^3-3u^2-u+1.$
(11; 4, 4, 3)	sixes connected; one single 4 surrounded by 5556; other single 4 surrounded by 5566 [*ab, ac, ad, ae, af, ag, bc, bj, bi, bh, bg, cd, cj, de, dj, ef, ek, ei, ej, fg, fk, gh, gk, hi, hk, ki, ij*] $u^7+0u^6+7u^5-6u^4+12u^3-2u^2-3u+0.$
(11; 4, 4, 3)	sixes connected; *each* single 4 surrounded by 5566 [*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, cj, ci, ch, de, dj, ef, ek, ej, fg, fk, gh, gi, gk, hi, ij, ik, kj*] $u^7+0u^6+7u^5-7u^4+14u^3+0u^2+0u+1.$
(11; 4, 4, 3)	sixes disconnected [*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, ci, ch, de, di, ef, ek, ej, ei, fg, fk, gh, gk, hi, hj, hk, ij, jk*] $u^7+0u^6+7u^5-8u^4+13u^3-3u^2-u+1.$
(11; 4, 5, 1, 1)	six and seven separate [*ab, ac, ad, ae, af, ag, ah, bc, bj, bi, bh, cd, cj, de, dk, dj, ef, ek, fg, fk, gh, gi, gk, hi, ij, ik, jk*] $u^7+0u^6+7u^5-7u^4+8u^3-5u^2-u+1.$
(11; 4, 5, 1, 1)	six and seven in contact [*ab, ac, ad, ae, af, ag, ah, bc, bk, bj, bh, cd, ck, de, dk, ef, ei, ej, ek, fg, fi, gh, gi, hj, hi, ij, jk*] $u^7+0u^6+7u^5-4u^4+11u^3+0u^2-2u+0.$
(11; 5, 2, 4)	[*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, cj, ci, ch, de, dk, dj, ef, ek, fg, fj, fk, gh, gi, gj, hi, ij, jk*] $u^7+0u^6+7u^5-u^4+12u^3+6u^2+3u+1.$
(11; 5, 3, 2, 1)	[*ab, ac, ad, ae, af, ag, ah, bc, bj, bi, bh, cd, cj, de, dj, ef, ek, ej, fg, fi, fk, gh, gi, hi, ki, kj, ij*] $u^7+0u^6+7u^5-5u^4+7u^3-3u^2+0u+1.$
(11; 6, 0, 5)	[*ab, ac, ad, ae af, ag, bc, bj, bi, bh, bg, cd, cj, de, dk, di,*

	dj, ef, ek, fg, fh, fi, fk, gh, hi, ij, ik]
	$u^7+0u^6+7u^5-3u^4+14u^3+8u^2+7u+5+u^{-1}$.
(11; 6, 1, 3, 1)	[*ab, ac, ad, ae, af, ag, ah, bc, bi, bh, cd, ci, de, dk, dj, di, ef, ek, fg, fk, gh, gi, gj, gk, hi, ij, jk*]
	$u^7+0u^6+7u^5-u^4+12u^3+6u^2+3u+1$.
(11; 6, 2, 2, 0, 1)	[*ab, ac, ad, ae, af, ag, ah, ai, bc, bk, bj, bi, cd, ck, de, dk, ef, ek, fg, fj, fk, gh, gj, hi, hj, ij, jk*]
	$u^7+0u^6+7u^5+0u^4+5u^3-2u^2+u+1$.
(11; 6, 3, 0, 1, 1)	[*ab, ac, ad, ae, af, ag, ah, ai, bc, bj, bi, cd, ck, cj, de, dk, ef, ek, fg, fk, gh, gk, hi, hj, hk, ij, jk*]
	$u^7+0u^6+7u^5+u^4+5u^3-2u^2+u+1$.
(11; 7, 2, 0, 0, 2)	[*ab, ac, ad, ae, af, ag, ah, ai, bc, bj, bi, cd, ck, cj, de, dk, ef, ek, fg, fk, gh, gk, hi, hk, ij, ik, jk*]
	$u^7+0u^6+7u^5+5u^4+8u^3+u^2+2u+1$.
(11; 9, 0, 0, 0, 0, 2)	[*ab, ac, ad, ae, af, ag, ah, ai, aj, bc, cd, de, ef, fg, gh, hi, ij, jb, bk, ck, dk, ek, fk, gk, hk, ik, jk*]
	$u^7+0u^6+7u^5+15u^4+25u^3+21u^2+13u+3$.
(12; 0, 12)	[*ab, ac, ad, ae, af, bc, bh, bg, bf, cd, ci, ch, de, dj, di, ef, ek, ej, fg, fk, gh, gl, gk, hi, hl, ij, il, jk, jl, kl*]
	$u^8+0u^7+8u^6-14u^5+39u^4-42u^3+12u^2+8u-2$.
(12; 2, 8, 2)	sixes in contact [*ab, ac, ad, ae, af, ag, bc, bj, bi, bh, bg, cd, cj, de, dk, dj, ef, el, ek, fg, fh, fl, gh, hi, hl, ij, ik, il, jk, kl*]
	$u^8+0u^7+8u^6-12u^5+31u^4-25u^3+5u^2+5u-1$.
(12; 2, 8, 2)	sixes separate [*ab, ac, ad, ae, af, ag, bc, bj, bi, bg, cd, ck, cj, de, dl, dk, ef, eh, el, fg, fh, gi, gh, ij, il, ih, jk, jl, kl, lk*]
	$u^8+0u^7+8u^6-12u^5+27u^4-25u^3+3u^2+4u-1$.
(12; 3, 6, 3)	sixes disconnected, but not mutually isolated [*ab, ac, ad, ae, af, ag, bc, bi, bh, bg, cd, cj, ci, de, dk, dj, ef, ek, fg, fl, fk, gh, gl, lh, lk, hi, hj, hk, ij, jk*]
	$u^8+0u^7+8u^6-11u^5+25u^4-19u^3+3u^2+3u-1$.
(12; 3, 6, 3)	sixes mutually isolated [*ab, ac, ad, ae, af, ag, bc, bi, bh, bg, cd, ci, de, dl, di, ef, ek, el, fg, fk, gh, gk, hi, hj, hk, il, ij, jl, jk, lk*]
	$u^8+0u^7+8u^6-11u^5+30u^4-15u^3+9u^2+5u-1$.
(12; 3, 6, 3)	sixes connected [*ab, ac, ad, ae, af, ag, bc, bi, bh, bg, cd, cl, ck, ci, de, dl, ef, el, fg, fj, fk, fl, gh, gj, hi, hj, ik, ij, jk, kl*]
	$u^8+0u^7+8u^6-9u^5+21u^4-16u^3-2u^2+3u+0$.
(12; 3, 7, 1, 1)	fours mutually isolated [*ab, ac, ad, ae, af, ag, ah, bc, bk, bj, bh, cd, ck, de, dl, dk, ef, el, fg, fi, fl, gh, gi, hj, hi, li, lj, lk, ij, jk*]
	$u^8+0u^7+8u^6-11u^5+21u^4-18u^3+4u^2+3u-1$.

(12; 3, 7, 1, 1) fours not mutually isolated [*ab, ac, ad, ae, af, ag, ah, bc, bi, bh, cd, cj, ci, de, dk, dj, ef, ek, fg, fl, fk, gh, gi, gl, hi ij, il, jl, jk, lk*]
$u^8+0u^7+8u^6-10u^5+23u^4-12u^3+6u^2+3u-1.$

(12; 4, 4, 4) [*ab, ac, ad, ae, af, ag, bc, bj, bi, bh, bg, cd, ck, cj, de, dk, ef, el, ei, ek, fg, fl, gh, gl, hi, hl, ij, ik, il, jk*]
$u^8+0u^7+8u^6-9u^5+21u^4-16u^3-2u^2+3u+0.$

(12; 4, 5, 2, 1) [*ab, ac, ad, ae, af, ag, ah, bc, bi, bh, cd, cj, ck, ci, de, dj, ef, el, ej, fg, fl, gh, gi, gk, gl, hi, ik, kj, kl, lj*]
$u^8+0u^7+8u^6-8u^5+21u^4-11u^3+0u^2+3u+0.$

(12; 4, 6, 1, 0, 1) [*ab, ac, ad, ae, af, ag, ah, al, bc, bk, bj, bl, cd, ck, de, dk, ef, ej, ek, fg, fi, fj, gh, gi, hl, hi, li, lj, jk, ij*]
$u^8+0u^7+8u^6-4u^5+13u^4-5u^3+u^2+2u+0.$

(12; 5, 2, 5) [*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, cj, ci, ch, de, dj, ef, el, ek, ej, fg, fl, gh, gi, gl, ih, ij, ik, il, kj, kl*]
$u^8+0u^7+8u^6-8u^5+24u^4-5u^3+7u^2+5u+0.$

(12; 5, 3, 3, 1) [*ab, ac, ad, ae, af, ag, ah, bc, bj, bi, bh, cd, cj, de, dj, ef, el, ek, ej, fg, fl, gh, gl, hi, hl, ij, ik, il, kj, kl*]
$u^8+0u^7+8u^6-7u^5+22u^4-u^3+9u^2+5u+0.$

(12; 5, 4, 1, 2) [*ab, ac, ad, ae, af, ag, ah, bc, bl, bk, bj, bi, bh, cd, cl, de, dk, dl, ef, ej, ek, fg, fj, gh, gi, gj, hi, kl, kj, ij*]
$u^8+0u^7+8u^6-4u^5+16u^4-5u^3-2u^2+u+0.$

(12; 5, 4, 2, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, bj, bk, bl, bi, cd, cj, de, dj, ef, ek, ej, fg, fk, gh, gl, gk, hi, hl, il, kj, kl*]
$u^8+0u^7+8u^6-4u^5+13u^4-5u^3+u^2+2u+0.$

(12; 5, 5, 0, 1, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, bj, bi, dc, cj, de, dk, dj, ef, ek, fg, fl, fk, gh, gl, hi, hj, hl, ij, jk, jl, kl*]
$u^8+0u^7+8u^6-6u^5+9u^4-7u^3+5u^2+u-1.$

(12; 6, 0, 6) [*ab, ac, ad, ae, af, ag, bc, bl, bk, bj, bg, cd, cl, de, dj, dk, ef, eh, ei, ej, fg, fh, gj, gi, gh, jk, ij, kl, hi, dl*]
$u^8+0u^7+8u^6+0u^5+19u^4+13u^3+17u^2+12u+5+u^{-1}.$

(12; 6, 4, 0, 0, 2) fives connected [*ab, ac, ad, ae, af, ag, ah, ai, bc, bj, bi, cd, cl, cj, de, dl, ef, ek, el, fg, fj, fk, gh, gj, hi, hj, ij, jl, jk, kl*]
$u^8+0u^7+8u^6-2u^5+7u^4-4u^3+4u^2+0u-1.$

(12; 6, 4, 0, 0, 2) fives disconnected; one pair of adjacent fives [*ab, ac, ad, ae, af, ag, ah, ai, ib, bc, cd, de, ef, fg, gh, hi, jk, cj, dj, ej, kl, ki, kb, kc, ke, kf, fl, gl, hl, hk*]
$u^8+0u^7+8u^6+u^5+9u^4-2u^3+3u^2+2u+0.$

(12; 6, 4, 0, 0, 2) fives mutually isolated [*ab, ac, ad, ae, af, ag, ah, ai, ib, bc, cd, de, ef, fg, gh, hi, ij, jc, jb, kg, kh, ki, kj, kc, kd, ke, kl, fl, gl, el*]
$u^8+0u^7+8u^6+u^5+9u^4-2u^3+3u^2+2u+0.$

(12; 7, 0, 4, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, ib, bc, cd, de, ef, fg, gh, hi, jk, kl, fj, gj, hj, ij, bj, bk, fk, fl, bl, cl, dl, el*]
$u^8+0u^7+8u^6+0u^5+19u^4+13u^3+17u^2+12u+5+u^{-1}$.

(12; 7, 2, 1, 0, 2) [*ab, ac, ad, ae, af, ag, ah, ai, ib, bc, cd, de, ef, fg, gh, hi, jk, kl, hk, ik, bk, ck, dk, fk, dj, fj, je, fl, gl, hl*]
$u^8+0u^7+8u^6+u^5+9u^4-2u^3+3u^2+2u+0$.

(12; 8, 0, 2, 0, 2) [*ab, ac, ad, ae, af, ag, ah, ai, ib, bc, cd, de, ef, fg, gh, hi, kb, kc, kd, ke, kf, kg, kh, kj, bj, hj, il, bl, jl, hl*]
$u^8+0u^7+8u^6+6u^5+18u^4+18u^3+22u^2+13u+5+u^{-1}$.

(12; 8, 2, 0, 0, 0, 2) [*ab, ac, ad, ae, af, ag, ah, ai, aj, ij, jb, bc, cd, de, ef, fg, gh, hi, kl, ki, kj, bk, bl, cl, dl, el, fl, gl, hl, il*]
$u^8+0u^7+8u^6+9u^5+15u^4+7u^3+5u^2-u-1$.

(12; 10, 0, 0, 0, 0, 0, 2) [*ab, ac, ad, ae, af, ag, ah, ai, aj, ak, bc, cd, de, ef, fg, gh, hi, ij, jk, kb, lb, lc, ld, le, lf, lg, lh, li, lj, lk*]
$u^8+0u^7+8u^6+21u^5+41u^4+45u^3+35u^2+15u+5+u^{-1}$.

(13; 1, 10, 2) [*ab, ac, ad, ae, af, ag, bc, bh, bg, cd, ci, ch, de, dj, di, ef, ek, ej, fg, fl, fk, gh, gl, hm, mi, mj, mk, ml, ij, jk, kl, hl, hi*]
$u^9+0u^8+9u^7-15u^6+48u^5-65u^4+49u^3-6u^2-9u+2$.

(13; 2, 8, 3) sixes connected [*ab, ac, ad, ae, af, ag, bc, bl, bk, bg, cd, cl, de, dh, dm, dl, ef, ei, eh, fg, fi, gk, gj, gi, ij, ih, lm, lk, km, kj, hj, mj, mh*]
$u^9+0u^8+9u^7-14u^6+42u^5-53u^4+37u^3+2u^2-6u+1$.

(13; 2, 8, 3) sixes disconnected [*ab, ac, ad, ae, af, ag, bc, bi, bh, bg, cd, cj, ci, de, dk, dj, ef, ek, fg, fl, fk, gh, gm, gl, lk, kj, lj, lm, hm, hi, mi, mj, ij*]
$u^9+0u^8+9u^7-14u^6+37u^5-51u^4+29u^3+u^2-5u+1$.

(13; 2, 9, 1, 1) [*ab, ac, ad, ae, af, ag, ah, hi, hb, hg, gi, gf, gm, fm, fe, fl, ed, el, dc, dl, kl, dk, bc, ck, cj, bi, bj, ij, mj, mi, mk, lm, jk*]
$u^9+0u^8+9u^7-14u^6+36u^5-46u^4+31u^3+u^2-5u+1$.

(13; 3, 7, 2, 1) has a 6-6-7 open chain [*ab, ac, ad, ae, af, ag, ah, bc, bi, bh, cd, cl, cm, ci, de, dl, ef, ek, el, fg, fj, fk, gh, gi, gj, hi, im, ij, lm, km, jm, kl, jk*]
$u^9+0u^8+9u^7-12u^6+38u^5-35u^4+23u^3-u^2-4u+1$.

(13; 3, 7, 2, 1) has a 6-7-6 open chain [*ab, ac, ad, ae, af, ag, ah, bc, bi, bh, cd, cj, ci, de, dm, dl, dj, ef, em, fg, fm, gh, gk, gl, gm, hi, hk, lm, kl, jl, jk, ki, ij*]
$u^9+0u^8+9u^7-11u^6+33u^5-33u^4+18u^3+6u^2-3u+0$.

(13; 3, 7, 2, 1) fours mutually isolated [*ab, ac, ad, ae, af, ag, ah, bc, bi, bh, cd, cj, ci, de, dk, dj, ef, ek, fg, fm, fl, fk, gh, gm, hi, hm, li, lj, lk, lm, mi, ij, jk*]
$u^9+0u^8+9u^7-13u^6+37u^5-41u^4+25u^3-2u^2-5u+1$.

(13; 4, 6, 1, 2) [*ab, ac, ad, ae, af, ag, ah, bc, bi, bm, bh, cd, ci, de, dj, di,*

ij, ik, il, im, ef, ek, ej, fg, fk, gh, gl, gk, jk, kl, lm, hm, hl]
$u^9+0u^8+9u^7-12u^6+27u^5-30u^4+17u^3-2u^2-3u+1.$

(13; 4, 6, 2, 0, 1) sixes isolated from the eight [*ab, ac, ad, ae, af, ag, ah, ai, jk, jl, kl, km, lm, bc, cd, de, ef, fg, gh, hi, ib, bk, bj, cj, dj, ej, el, fl, fm, gm, hm, im, ik*]
$u^9+0u^8+9u^7-10u^6+29u^5-15u^4+21u^3+0u^2-2u+1.$

(13; 4, 6, 2, 0, 1) both sixes in contact with the eight [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, ib, jk, kl, lm, bk, bl, bm, cm, dm, em, el, fl, fk, fj, gj, hj, ij, ik*]
$u^9+0u^8+9u^7-6u^6+24u^5-15u^4+6u^3+6u^2+0u+0.$

(13; 4, 7, 0, 1, 1) eight and seven in contact [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, ib, jk, kl, lm, bj, bk, bl, bm, cm, dm, em, el, fl, fk, gk, gj, hj, ij*]
$u^9+0u^8+9u^7-6u^6+24u^5-15u^4+6u^3+6u^2+0u+0.$

(13; 4, 7, 0, 1, 1) eight and seven separate [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, ib, jk, jl, jm, kl, lm, bj, cj, ck, dk, ek, el, fl, fm, gm, hm, hj, ij*]
$u^9+0u^8+9u^7-11u^6+22u^5-21u^4+16u^3-4u^2-3u+1.$

(13; 5, 2, 6) [*ab, ac, ad, ae, af, ag, bc, bj, bi, bh, bg, cd, cj, de, el, dk, dj, ef, el, fg, fh, fl, gh, hi, ij, jk, kl, lh, hm, im, jm, km, lm*]
$u^9+0u^8+9u^7-11u^6+39u^5-28u^4+25u^3+4u^2-2u+1.$

(13; 5, 6, 0, 1, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai aj, bc, cd, de, ef, fg, gh, hi, ij, jb, kl, lm, bl, cl, cm, dm, em, fm, fl, gl, gk, hk, ik, jk, jl*]
$u^9+0u^8+9u^7-3u^6+14u^5-7u^4+6u^3-u^2-2u+0.$

(14; 0, 12, 2) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, bh, ch, ci, di, dj, ej, ek, fk, fl, gl, gm, bm, hi, ij, jk, kl, lm, mh, in, jn, kn, ln, mn, hn*]
$u^{10}+0u^9+10u^8-18u^7+64u^6-118u^5+160u^4-89u^3$
$-3u^2+16u-3.$

(14; 1, 10, 3) [*ab, ac, ad, ae, af, ag, bh, bi, ci, cj, dj, dk, ek, el, fl, fm, gm, gh, bc, cd, de, ef, fg, bg, hi, ij, jk, kl, lm, mh, jl, hn, in, jn, ln, mn*]
$u^{10}+0u^9+10u^8-17u^7+60u^6-105u^5+126u^4-64u^3$
$-5u^2+12u-2.$

(14; 2, 8, 4) fours disconnected [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, bh, ch, ci, cj, dj, dk, ek, el, fl, fm, gm, gh, hi, ij, jk, kl, lm, mh, in, jn, kn, ln, mn, hn*]
$u^{10}+0u^9+10u^8-16u^7+60u^6-94u^5+113u^4-49u^3$
$-2u^2+10u-2.$

(14; 2, 8, 4) fours connected [*ah, ai, aj, ak, am, an, hi, ij, jk, km, mn, nh, gm, gn, gk, gh, bg, bh, ch, ci, di, dj, ej, ek, fk, fg, bc, cd, de, ef, fb, lb, lc, ld, le, lf*]

$u^{10}+0u^9+10u^8-15u^7+58u^6-85u^5+97u^4-38u^3$
$-7u^2+8u-1.$

(14; 2, 9, 2, 1) sixes in contact [*ab, ac, ad, ae, af, ag, ah, bc, bi, bn, bm, bh, in, jn, kn, ln, mn, cd, de, ef, fg, gh, ij, jk, kl, lm, ci, di, dj, ej, ek, fk, fl, gl, gm, hm*]
$u^{10}+0u^9+10u^8-16u^7+53u^6-81u^5+95u^4-38u^3-9u^2$
$+7u-1.$

(14; 2, 9, 2, 1) sixes disconnected; fives doubly connected [*ab, ac, ad ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, bj, bn, cn, dn, dk, ek, el, fl, fm, gm, hm, hi, ij, jk, kl, il, jl, lm, im, kn, jn*]
$u^{10}+0u^9+10u^8-16u^7+50u^6-82u^5+85u^4-34u^3-6u^2$
$+7u-1.$

(14; 2, 9, 2, 1) sixes disconnected; fives simply connected [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, bj, cj, dj, dk, ek, el, fl, fm, gm, hm, hi, ij, jk, kl, lm, mi, in, jn, kn, ln, mn*]
$u^{10}+0u^9+10u^8-16u^7+57u^6-87u^5+92u^4-53u^3+0u^2$
$+10u-2.$

(14; 2, 10, 0, 2) [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, ci, cj, dj, dk, ek, el, fl, fm, gm, gi, hi, ij, jk, kl, lm, mi, in, jn, kn, ln, mn*]
$u^{10}+0u^9+10u^8-15u^7+54u^6-71u^5+71u^4-41u^3+6u^2$
$+9u-2.$

(14; 3, 7, 3, 1) fours mutually isolated [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, ci, cj, ck, dk, ek, el, fl, gl, gm, gn, hn, hi, in, jn, jm, km, kl, ij, jk, lm, mn*]
$u^{10}+0u^9+10u^8-15u^7+50u^6-76u^5+73u^4-30u^3-5u^2$
$+6u-1.$

(14; 3, 7, 3, 1) pair of connected fours [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, bj, cj, dj, di, dk, ek, el, fl, fm, gm, hm, hn, hi, ij, in, il, kl, kn, ln, lm, mn*]
$u^{10}+0u^9+10u^8-13u^7+47u^6-58u^5+60u^4-15u^3-11u^2$
$+3u+0.$

(14; 3, 8, 2, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, bi, ij, jk, kl, lm, mn, nj, jm, jl, bj, bk, ck, dk, ek, el, fl, fm, gm, gn, hn, in*]
$u^{10}+0u^9+10u^8-14u^7+43u^6-52u^5+51u^4-25u^3-u^2$
$+5u-1.$

(14; 3, 9, 0, 1, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, bi, bj, bk, ck, cl, dl, el, em, fm, gm, gn, hn, in, ij, jk, kl, lm, mn, nj, km, jm*]

$u^{10}+0u^9+10u^8-15u^7+38u^6-51u^5+52u^4-25u^3-u^2$
$+5u-1.$

(14; 4, 8, 0, 1, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, aj, bc, cd, de, ef, fg, gh, hi, ij, jb, bk, bn, cn, dn, dl, el, em, fm, gm, hm, hl, il, ik, jk, kl, lm, ln, kn*]
$u^{10}+0u^9+10u^8-9u^7+25u^6-24u^5+22u^4-8u^3-3u^2$
$+2u+0.$

(14; 6, 0, 8) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, hi, hj, hk, hl, hm, hn, ij, jk, kl, lm, mn, ni, bj, cj, ck, cl, dl, el, em, en, fn, gn, gi, gj*]
$u^{10}+0u^9+10u^8-12u^7+58u^6-48u^5+85u^4+9u^3+23u^2$
$+20u+5+u^{-1}.$

(14; 6, 3, 4, 0, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, aj, bc, cd, de, ef, fg, gh, hi, ij, jb, bk, bl, bm, cm, dm, em, el, en, fn, gn, hn, hl, hk, ik, jk, kl, lm, ln*]
$u^{10}+0u^9+10u^8-6u^7+32u^6-11u^5+32u^4+8u^3-3u^2$
$+0u+0.$

(15; 0, 12, 3) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, bh, bi, ci, cj, dj, dk, ek, el, fl, fm, gm, gh, hi, ij, jk, kl, lm, mh, ho, io, jo, ko, kn, ln, mn, hn, no*]
$u^{11}+0u^{10}+11u^9-20u^8+78u^7-170u^6+291u^5-284u^4$
$+128u^3+18u^2-25u+4.$

(15; 1, 11, 2, 1) four in contact with a six [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, ci, cj, dj, dk, ek, el, fl, fm, gm, gn, hn, hi, ij, jk, kl, lm, mn, ni, io, jo, ko, lo, mo, no*]
$u^{11}+0u^{10}+11u^9-19u^8+74u^7-138u^6+224u^5-200u^4$
$+78u^3+10u^2-17u+3.$

(15; 1, 11, 2, 1) four not in contact with a six [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, ci, cj, dj, dk, ek, el, fl, fm, gm, gn, hn, hi, ij, jk, kl, lm, mn, ni, jn, jo, ko, lo, mo, no*]
$u^{11}+0u^{10}+11u^9-19u^8+70u^7-136u^6+207u^5-184u^4$
$+64u^3+16u^2-14u+2.$

(15; 2, 8, 5) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, bh, bi, ci, cj, ck, dk, ek, el, fl, gl, gm, gh, hi, ij, jk, kl, lm, mh, jn, kn, ln, mn, oh, oi, oj, on, om*]
$u^{11}+0u^{10}+11u^9-18u^8+72u^7-137u^6+214u^5-173u^4$
$+55u^3+14u^2-13u+2.$

(15; 2, 9, 3, 1) pair of sixes plus an isolated six [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, ci, cj, ck, dk, dl, el, fl, fm, gm, gn, go, ho, hi, ij, jk, kl, lm, mn, no, oi, jo, jn, kn, km*]
$u^{11}+0u^{10}+11u^9-18u^8+63u^7-120u^6+170u^5-131u^4$
$+29u^3+15u^2-8u+1.$

(15; 2, 9, 3, 1) mutually isolated sixes [*ab, ac, ad, ae, af, ag, ah, bc, cd,*

de, ef, fg, gh, hb, bi, bj, cj, dj, dk, ek, el, fl, fm, gm, hm, hn, hi, ij, jk, kl, lm, mn, ni, ln, io, jo, ko, lo, no]
$u^{11}+0u^{10}+11u^9-18u^8+70u^7-135u^6+188u^5-166u^4$
$+59u^3+13u^2-13u+2.$

(15; 2, 10, 1, 2) [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, bj, bk, bl, cl, dl, dm, em, en, fn, fo, go, gi, hi, lm, mn, no, oi, kl, km, kn, jn, jo, ij, jk*]
$u^{11}+0u^{10}+11u^9-18u^8+64u^7-119u^6+178u^5-125u^4$
$+37u^3+15u^2-9u+1.$

(15; 2, 11, 0, 1, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, ib, bj, ck, dk, dl, el, em, fm, fn, gn, go, ho, io, ij, bk, jk, kl, lm, mn, no, oj, jl, jm, jn*]
$u^{11}+0u^{10}+11u^9-18u^8+57u^7-90u^6+130u^5-104u^4$
$+27u^3+10u^2-7u+1.$

(15; 3, 8, 2, 2) [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bn, bo, bi, ci, cj, dj, dk, ek, el, fl, fm, gm, hm, hn, ij, jk, kl, lm, mn, io, jo, ko, lo, mo, no*]
$u^{11}+0u^{10}+11u^9-17u^8+63u^7-99u^6+147u^5-107u^4$
$+28u^3+12u^2-7u+1.$

(15; 5, 5, 3, 1, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, ib, bk, kl, km, kn, ko, kg, kj, bl, lm, mn, no, go, gj, bj, ij, lc, hj, fo, eo, en, em, dm, dl*]
$u^{11}+0u^{10}+11u^9-13u^8+51u^7-56u^6+83u^5-45u^4+u^3$
$+8u^2-2u+0.$

(15; 6, 4, 3, 1, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, aj, bc, cd, de, ef, fg, gh, hi, ij, bj, jk, jl, jo, kl, km, kd, kn, kg, ko, go, lm, dm, dn, gn, ho, io, bl, bm, cm, en, fn*]
$u^{11}+0u^{10}+11u^9-9u^8+40u^7-30u^6+52u^5-20u^4-10u^3$
$+3u^2+0u+0.$

(15; 6, 5, 2, 1, 0, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, aj, ck, bc, cd, de, ef, fg, gh, hi, ij, jk, kb, bm, bn, cn, dn, en, em, eo, fo, go, ho, hm, hl, il, jl, kl, km, lm, mn, mo*]
$u^{11}+0u^{10}+11u^9-6u^8+33u^7-14u^6+41u^5-5u^4+u^3$
$+4u^2+0u+0.$

(16; 0, 12, 4) sixes mutually isolated [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, fl, gl, gh, bh, bi, ci, ij, jk, kl, ih, hj, hk, hl, lm, mn, np, pi, fm, em, en, dn, dp, cp, km, om, on, op, jp, oj, ok*]
$u^{12}+0u^{11}+12u^{10}-22u^9+93u^8-232u^7+468u^6-639u^5$
$+541u^4-180u^3-47u^2+41u-6.$

(16; 0, 12, 4) two mutually isolated pairs of sixes [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, gh, hb, hi, ib, ij, bj, cj, jk, ck, dk, kl, dl, el, lm, em, fm, fn, gn, mn, nh, op, om, on, oh, oi,*

pi, pj, pk, pl, pm]
$u^{12}+0u^{11}+12u^{10}-22u^{9}+93u^{8}-218u^{7}+448u^{6}-614u^{5}$
$+487u^{4}-144u^{3}-46u^{2}+35u-5.$

(16; 0, 14, 0, 2) [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, ij, ik, il, im, in, io, ip, jk, kl, lm, mn, no, op, pj, bj, bk, ck, cl, dl, dm, em, en, fn, fo, go, gp, hp, hj*]
$u^{12}+0u^{11}+12u^{10}-22u^{9}+91u^{8}-187u^{7}+370u^{6}-499u^{5}$
$+380u^{4}-93u^{3}-33u^{2}+26u-4.$

(16; 1, 10, 5) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, gh, gi, gj, fh, hi, ij, cj, bj, oh, oi, ok, ol, om, on, ik, kl, lm, mn, nh, fn, en, em, dm, jk, dl, cp, jp, kp, lp, dp*]
$u^{12}+0u^{11}+12u^{10}-21u^{9}+91u^{8}-212u^{7}+415u^{6}-536u^{5}$
$+409u^{4}-115u^{3}-41u^{2}+29u-4.$

(16; 1, 11, 3, 1) sixes connected [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, bi, hi, ij, bj, cj, gk, hk, ik, jk, fl, gl, kl, em, fm, lm, dn, en, mn, jo, co, do, no, pj, pk, pl, pm, pn, po*]
$u^{12}+0u^{11}+12u^{10}-21u^{9}+88u^{8}-184u^{7}+351u^{6}-440u^{5}$
$+317u^{4}-72u^{3}-32u^{2}+21u-3.$

(16; 1, 11, 3, 1) sixes disconnected [*ab, ac, ad, ae, af, ag, ah, bc, cd, de, ef, fg, gh, hb, hi, bi, ci, gj, hj, ij, jk, ik, ck, cl, kl, dl, lm, dm, em, mn, en, fn, no, fo, go, jo, pj, pk, pl, pm, pn, po*]
$u^{12}+0u^{11}+12u^{10}-21u^{9}+88u^{8}-191u^{7}+364u^{6}-458u^{5}$
$+328u^{4}-79u^{3}-35u^{2}+22u-3.$

(16; 2, 8, 6) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, bj, cj, dj, ij, di, ei, hi, eh, fh, gh, bk, gk, kl, bl, jl, lm, jm, im, mn, in, hn, ok, go, ho, no, po, pn, pm, pl, pk*]
$u^{12}+0u^{11}+12u^{10}-20u^{9}+88u^{8}-184u^{7}+363u^{6}-422u^{5}$
$+287u^{4}-60u^{3}-37u^{2}+18u-2.$

(16; 2, 11, 1, 1, 1) [*ab, ac, ad, ae, af, ag, ah, ai, bc, cd, de, ef, fg, gh, hi, ib, dj, cj, bj, kj, bk, lk, il, hl, bl, km, lm, hm, gm, kn, mn, gn, fn, ko, no, fo, eo, po, pe, pd, pj, pk*]
$u^{12}+0u^{11}+12u^{10}-20u^{9}+78u^{8}-137u^{7}+247u^{6}-281u^{5}$
$+174u^{4}-15u^{3}-23u^{2}+9u-1.$

(16; 4, 8, 2, 1, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, aj, bc, cd, de, ef, fg, gh, hi, ij, jb, hk, ik, jk, cm, dm, em, hl, gl, fl, kn, jn, bn, no, bo, co, mo, po, pm, pe, pl, ph, pk, pn, el*]
$u^{12}+0u^{11}+12u^{10}-16u^{9}+65u^{8}-85u^{7}+148u^{6}-132u^{5}$
$+67u^{4}+13u^{3}-11u^{2}+2u+0.$

(16; 6, 5, 3, 1, 0, 0, 1) [*ab, ac, ad, ae, af, ag, ah, ai, aj, ak, bc, cd, de, ef, fg, gh, hi, ij, jk, kb, jl, kl, bl, lm, bm, cm, dm, dn, en, fn, gn, go, ho, io, jo, po, pj, pl, pm, pd, pn, pg*]
$u^{12}+0u^{11}+12u^{10}-11u^{9}+50u^{8}-43u^{7}+86u^{6}-50u^{5}$
$+22u^{4}+14u^{3}-3u^{2}+0u+0.$

(17; 0, 12, 5) [*ab, ac, ad, ae, af, ag, bc, cd, de, ef, fg, gb, ch, ci, cj, bh, hi, ij, dj, fl, kl, lm, ik, il, im, hm, jk, en, fn, ln, kn, go, fo, lo, mo, po, pg, pb, ph, pm, eq, dq, jq, kq, nq*]

$u^{13}+0u^{12}+13u^{11}-24u^{10}+108u^9-259u^8+619u^7$
$-1039u^6+1152u^5-686u^4+92u^3+98u^2-39u+4.$

3. **Further special results concerning regular maps.** In addition to the preceding tabulated results, it is possible to obtain a limited amount of information by use of linear difference equations with constant coefficients. We illustrate by deducing the number of ways in which a proper ring of m regions can be colored from λ available colors: Denote the required number by $F_m(\lambda)$. Then it is easy to see that

$$F_m(\lambda) = (\lambda - 2)F_{m-1}(\lambda) + (\lambda - 1)F_{m-2}(\lambda). \tag{3.1}$$

For, if a, b, c denote three consecutive regions of the ring of m regions, $(\lambda-2)F_{m-1}(\lambda)$ is equal to the number of ways the ring can be colored in such a way that a and c are colored differently, while $(\lambda-1)F_{m-2}(\lambda)$ is the number of ways the ring can be colored so that a and c are colored alike. Now (3.1) is a second order linear difference equation with respect to m. We proceed to solve it under the appropriate initial conditions

$$F_2(\lambda) = \lambda(\lambda - 1), \qquad F_3(\lambda) = \lambda(\lambda - 1)(\lambda - 2). \tag{3.2}$$

The characteristic equation is $\rho^2-(\lambda-2)\rho-(\lambda-1)=0$, which has roots $\rho_1=(\lambda-1)$ and $\rho_2=-1$. It follows that

$$F_m(\lambda) = A(\lambda - 1)^m + B(- 1)^m, \tag{3.3}$$

where A and B are independent of m. Substituting successively $m=2$ and $m=3$, we find from (3.2) that $A=1$ and $B=\lambda-1$. Hence

$$F_m(\lambda) = (\lambda - 1)^m + (\lambda - 1)(- 1)^m. \tag{3.4}$$

This result was also obtained by Whitney by other methods (cf. Whitney [2, p. 691]) and leads at once to the following special result on regular maps:

THEOREM I. *The chromatic polynomial of a regular map P_n consisting of a proper ring of $n-2$ regions together with an interior and exterior region is given by*

$$\begin{aligned} P_n(\lambda) &= \lambda[(\lambda - 2)^{n-2} + (- 1)^{n-2}(\lambda - 2)] \\ &\quad + \lambda(\lambda - 1)[(\lambda - 3)^{n-2} + (- 1)^{n-2}(\lambda - 3)]. \end{aligned} \tag{3.5}$$

Proof. The exterior and interior region can be assigned the same color in λ ways. After this has been done, there are $\lambda-1$ colors available for the ring. Hence, using formula (3.4) with $m=n-2$ and λ replaced by $\lambda-1$, we find that the number of ways in which P_n can be colored so as to give the *same* color to the interior and exterior regions is $\lambda[(\lambda-2)^{n-2}+(-1)^{n-2}(\lambda-2)]$. A similar argument shows that the number of ways in which P_n can be colored so as to give *different* colors to the interior and exterior regions is $\lambda(\lambda-1)[(\lambda-3)^{n-2}+(-1)^{n-2}(\lambda-3)]$. Formula (3.5) results from the addition of these two quantities.

This formula (3.5) may be used to check the polynomials given in the preceding table for the maps (6; 6), (7; 5, 2), (8; 6, 0, 2), (9; 7, 0, 0, 2), and so on.

It will be observed from (3.5) that $\lim_{n\to\infty}[P_n(\lambda)]^{1/n}=\lambda-2$. This "asymptotic result" does not depend upon the explicit formula (3.5) but only on the result $\lim_{m\to\infty}[F_m(\lambda)]^{1/m}=\lambda-1$, which follows directly from (3.3) and the fact that $A>0$. Hence, in this sense, the asymptotic behavior of a family of chromatic polynomials which can be deduced in this way from linear difference equations with constant coefficients depends essentially only on the root of largest absolute value of the characteristic equation. Making use of this idea, we obtained the following result with regard to a considerably more complicated family of maps:

THEOREM II. *Let P_{2n} denote the map (regular for $n\geqq 5$) consisting of an "interior" $(n-1)$-sided region surrounded by a proper $(n-1)$-ring of pentagons, which in turn is surrounded by another proper $(n-1)$-ring of pentagons, the "exterior" region being an $(n-1)$-sided region*(7). *Then* $\lim_{n\to\infty}[P_{2n}(4)]^{1/2n}=(r)^{1/2}=1.353\cdots$, *where r is the (only) real root of the equation $\rho^3+\rho^2-3\rho-4=0$.*

The proof will be omitted inasmuch as the slight importance of the theorem hardly justifies the inclusion of its rather involved proof. Still another theorem of this type, whose proof will likewise be omitted, is the following:

THEOREM III. *Let $P_{5n+2}(n\geqq 2)$ denote the regular map consisting of an "interior" 5-sided region surrounded by n distinct proper 5-rings (of which the first and the last are rings of five-sided regions, the others are rings of six-sided regions) and an "exterior" five-sided region*(8). *Then* $\lim_{n\to\infty}[P_{5n+2}(4)]^{1/(5n+2)}=[3+5^{1/2}]^{1/5}=1.393\cdots$. *In fact, in this case, we can make the more explicit statement that* $P_{5n+2}(4)=12\ [(5-2\cdot 5^{1/2})(3+5^{1/2})^n+(5+2\cdot 5^{1/2})(3-5^{1/2})^n]$.

4. **Non-regular maps of triple vertices.** We now consider proper maps (that is, maps without isthmuses) with triple vertices only. The main result of this section depends on the following theorem, which is also of importance for other reasons.

Fixing attention on some region U of a map P_n, let us define a certain set F of vertices of P_n by saying that a vertex B belongs to the set F if, and only if, it is not a vertex of U but is connected to U by a boundary line having B for one end point and having for its other end point a vertex of U.

(7) For $n=5, 6, 7, 8$, P_{2n} is illustrated in §2 by (10; 2, 8), (12; 0, 12), (14; 0, 12, 2), (16; 0, 14, 0, 2) respectively.

(8) For $n=2, 3$ the map is illustrated in §2 by (12; 0, 12) and (17; 0, 12, 5). The map (7; 5, 2) can also be considered as belonging to the family for $n=1$, although in the above definition it was convenient to make the restriction $n\geqq 2$.

THEOREM I. *If every region abutting at a vertex of the set F (assumed not vacuous) also abuts the region U just once, then P_n contains at least one three-sided region abutting the region U.*

Proof. Choose any vertex $B_1 \in F$, so that there is a boundary line L_1 having one end at B_1 and the other end abutting U. Let S_1 and R_1 be the two regions having the side L_1 in common. Let S_2 be the third region abutting B_1 and hence, by hypothesis, also abutting U. If the (not necessarily proper) 3-ring US_1S_2 contains only R_1 on one side, which we hereafter call the "inside," then R_1 is three-sided and the theorem is true. Hence we limit attention to the case when the ring US_1S_2 has more than one region completely on the inside. The part of the boundary of S_2 which lies inside this ring must have at least one vertex not on U other than B_1. Otherwise, since L_1 is the complete inside boundary of S_1, R_1 would have to abut U more than once, contrary to hypothesis. Let B_2 therefore be the vertex on the inside boundary of S_2 nearest to U but not on U; and denote by L_2 the boundary line of S_2 abutting B_2 and U. Let R_2 be the other region which has L_2 as a boundary line. Since $B_2 \in F$ by definition of F, the third region abutting B_2, which we call S_3, must by hypothesis also abut U. The ring US_2S_3 contains fewer regions inside than US_1S_2, since S_3 is obviously completely inside US_1S_2 but is not completely inside US_2S_3. If the ring US_2S_3 contains only R_2, then R_2 is a three-sided region, and the theorem is true. Otherwise we repeat the process and obtain a ring US_3S_4 which contains still fewer regions. Since the map has only a finite number of regions, we must eventually find a ring US_kS_{k+1} which has only the one region R_k in its interior and this region clearly has three sides one of which is a side of U.

The following theorem is an almost obvious corollary of the preceding:

THEOREM II. *Let P_n $(n \geqq 3)$ be a map which contains a region U against which each of the other $n-1$ regions abut just once; then*

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3}. \tag{4.1}$$

Proof. The hypothesis of Theorem I is always satisfied by a map P_n of the stated type, at least for $n \geqq 4$. Moreover, it is obvious that we always get another map P_{n-1} of the same kind (with one less region) whenever we erase a side of a three-sided region abutting on U. By Principle (1.2) of Chapter I, we have $P_n(\lambda) = (\lambda - 3)P_{n-1}(\lambda)$ for $n \geqq 4$, while for $n = 3$ we obviously have $P_3(\lambda) = \lambda(\lambda-1)(\lambda-2)$. Hence for a map of this type (4.1) must hold.

Thus any two maps, each of which has the same number of regions and satisfies the hypothesis of Theorem II, must be chromatically equivalent. That they need not be topologically equivalent is clear from the maps illustrated in figure 16. In fact, one of these maps contains two five-sided regions; the other contains none. Each has a total of seven regions.

The theorems of this section suggest a species of induction which can certainly be used for the numerical computation of the chromatic polynomials as well as for the proof of some of their properties. The process may be explained as follows:

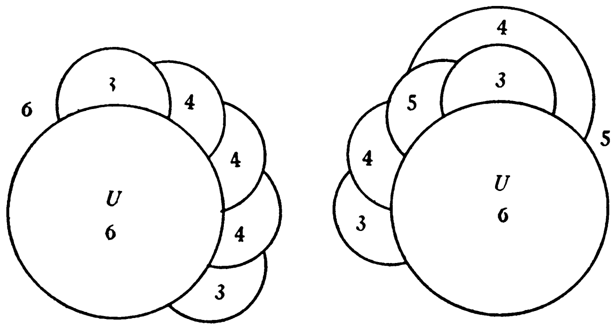


FIG. 16

Let P_n^k denote a map containing a certain k-sided region U upon which we fix attention. If the hypothesis of Theorem I is not fulfilled, we can select a vertex $B \in F$, such that one of the regions abutting at B does not have contact with U. We then "twist" the boundary line that connects B to U (cf. footnote 4) and apply Fundamental Principle (1.3) of Chapter I. The result is an equation of the form

$$P_n^k(\lambda) = P_n^{k+1}(\lambda) + P_{n-1}(\lambda) - P_{n-1}^*(\lambda), \tag{4.2}$$

where P_n^{k+1} has a region U with one more vertex than the region U in P_n^k. The same process can be repeated on the map P_n^{k+1} leading thus to a P_n^{k+2}, and so on. Eventually we arrive at $P_n{}^l$, $k \leqq l \leqq n-1$, for which the hypothesis of Theorem I (in an extreme case, the hypothesis of Theorem II) will hold. We can then apply fundamental principle (1.2) of Chapter I, obtaining

$$P_n^l(\lambda) = (\lambda - 3)\overline{P}_{n-1}(\lambda). \tag{4.3}$$

Thus, by equations of the type (4.2) and (4.3), we can express the original $P_n^k(\lambda)$ entirely in terms of chromatic polynomials of degree $n-1$, which may be assumed to be known.

As a simple application of this inductive process, the reader can prove by this method the known theorem (cf. Birkhoff [4]) that the first two terms in the Q polynomial for P_n^k are u^{n-4} and $0 \cdot u^{n-5}$ respectively, provided that the map has no proper 2-rings ($n \geqq 4$). It is merely necessary to observe that the process leading from P_n^k to P_n^{k+1} does not introduce 2-rings. A modification of this process is actually used in Chapter III to prove far reaching results, of which the above may be considered a primitive example. The modification involves the replacement of the region U by a multiple vertex.

The process is, however, not now available for proving the four-color theorem. This unfortunate circumstance is due, of course, to the negative term in (4.2), namely $-P^*_{n-1}(\lambda)$. Perhaps, if we had a sufficiently sharp inequality involving $P_{n-1}(\lambda)$ and $P^*_{n-1}(\lambda)$, this difficulty might be overcome. The inequality

$$-\lambda(\lambda-3)P_n^{k+1}(\lambda) \leqq (\lambda-3)P_{n-1}(\lambda) - (\lambda-3)(\lambda-1)P^*_{n-1}(\lambda) \leqq P_n^{k+1}(\lambda),$$

which is to be proved in Chapter V and is valid for positive integral values of λ (together with all values of $\lambda \geqq 5$), is not nearly sharp enough. It is possible that it might be sharpened if we were to use a hypothesis to the effect that no region in P_n^k has more than k sides.

5. **Maps with multiple vertices.** We close this chapter with the modified formulation of the results of the preceding section which will be immediately applicable in the next chapter.

We think of the k-sided region U as having been shrunk to a point V, which is a vertex of the map of multiplicity k. At the same time we set $m=n-1$. We deal exclusively in this section with maps P_m^k that can be so obtained, that is, maps of regions, whose closures are simply connected, having one vertex V of multiplicity k, but all other vertices of multiplicity three. The set F of vertices is defined so: $B \in F$ if, and only if, $B \not\equiv V$ and there is a boundary line having B and V for its end points. The modified forms of the two theorems of the preceding section can now be immediately written out.

THEOREM I. *If every region abutting at a vertex of the set F (assumed not empty) also abuts at V, the map P_m^k contains at least one two-sided region abutting at V.*

THEOREM II. *If every region of the map abuts at V just once (so that the multiplicity of V is m), then the chromatic polynomial of the map is*

$$P_m^m(\lambda) = \lambda(\lambda-1)(\lambda-2)^{m-2}. \tag{5.1}$$

Results of this character seem to have been known in somewhat different form by Whitney for some time. In fact, his formula

$$\delta_m = 3\cdot 2^m$$

(Whitney [4, p. 212]) is really a special case of (5.1) with $\lambda=4$.

CHAPTER III. THE EXPANSION OF THE CHROMATIC POLYNOMIALS IN POWERS OF $\lambda-2$

1. **A conjectured asymptotic formula.** A simple rational function of λ, namely $(\lambda-2)^2/(\lambda-1)$, turns out to be of fundamental importance in the rigorous deduction of certain inequalities satisfied by the coefficients of the chromatic polynomials written in powers of $\lambda-2$. It seems desirable therefore

to give the loose argument which led to this function in the first place. An attempt was made to find a simple "asymptotic formula" for chromatic polynomials of maps not containing too many of the known reducible configurations. The argument follows:

The number of contacts in a map of n simply connected regions, triple vertices, and without proper 2-rings, is $3n-6$ (cf. Birkhoff [4, p. 3]). Hence, if n is large, the average number of contacts per region is nearly 3. Thus, if we build up a map by adding successive regions to it, keeping the partially constructed map simply connected at each step and coloring it as we go along, each new region R, which we add, will (on the average) touch three of the regions already there. Call these regions A, B, and C, and assume that there is contact between A and B, and between B and C. A and B cannot have the same color, nor can C have the color of B. But the probability that C has the color of A is $1/(\lambda-1)$, and the probability that it does not have the color of A is $(\lambda-2)/(\lambda-1)$. In the first case R may be colored in $\lambda-2$ ways; in the second case in $\lambda-3$ ways. Hence, on the average R may be colored in

$$\frac{1}{\lambda-1}(\lambda-2)+\frac{\lambda-2}{\lambda-1}(\lambda-3)=\frac{(\lambda-2)^2}{\lambda-1}$$

ways.

From the fact that no map with at least one triple vertex can be colored in 0, 1 or 2 colors and only maps of even-sided regions can be colored in 3 colors, we assume the factors λ, $(\lambda-1)$, $(\lambda-2)$, $(\lambda-3)$. The conjectured asymptotic formula for the number of ways a map of n regions may be colored in λ colors is therefore

$$P_n(\lambda)\sim\lambda(\lambda-1)(\lambda-2)(\lambda-3)\left[\frac{(\lambda-2)^2}{\lambda-1}\right]^{n-4}. \tag{1.1}$$

The exponent $n-4$ is chosen corresponding to the total of n factors, one for each of the n regions.

The fact that this formula gives the number of ways the dodekahedron can be colored in 4 colors with a discrepancy of less than .27 appears to be an accident. If the formula has any significance at all, it is merely to the effect that

$$[P_n(\lambda)]^{1/n} \text{ is approximately equal to } \frac{(\lambda-2)^2}{\lambda-1} \tag{1.2}$$

for maps with a large number n of regions. It is seen very definitely that this is not true for the maps P_n of Theorem I, §3 of the last chapter. But these maps are of very special type, having a large number of four-sided regions. On the other hand, Theorems II and III of the same section seem to confirm the conjecture (as regards $\lambda=4$, at least), where we get the limits $1.353\cdots$ and $1.393\cdots$, both of which are reasonably close to the conjectured value of $1.333 \cdots = (\lambda-2)^2/(\lambda-1)\,|_{\lambda=4}$. These maps, too, have reducible configurations, but not of such an elementary type as that presented by the four-sided region. It is felt that formula (1.2) is likely to be more valid for maps with no reducible configurations or, at least, with only the reducible configurations of the more complicated types.

2. **Introductory remarks.** Unless otherwise stated, we are concerned throughout §§2–6 of this chapter with proper maps P^k_{n+3} of $n+3$ regions, whose closures are simply connected, with one vertex V of multiplicity k ($k \geqq 2$) and all other vertices triple. In case $k=2$ the point V is a vertex only by special convention. Actually it is an ordinary point on a boundary line of the map. Its exact location will not be subject to doubt when this case is met.

We use the following notation:

$$x = \lambda - 2, \tag{2.1}$$

$$Q^k_n(x) = \frac{P^k_{n+3}(\lambda)}{\lambda(\lambda-1)(\lambda-2)}, \tag{2.2}$$

where $Q^k_n(x)$ is obviously a polynomial of degree n in x with leading coefficient equal to one. It is convenient to write the polynomial as follows:

$$Q^k_n(x) = \sum_{h=0}^{n} (-1)^h a_h x^{n-h}. \tag{2.3}$$

We do not assume that all the a's are non-negative, though this will turn out to be the case (cf. Birkhoff [4, p. 10]). We also set

$$R = \frac{(\lambda-2)^2}{\lambda-1} = \frac{x^2}{1+x}, \tag{2.4}$$

$$U^k_n(x) = x^{k-3}R^{n-k+3} = x^{k-3}\left\{\frac{x^2}{1+x}\right\}^{n-k+3}. \tag{2.5}$$

It will be convenient also to have recorded here the expansion of $U^3_n(x) = R^n$ in descending powers of x. The binomial theorem gives

$$U^3_n(x) = R^n = x^n\left(1+\frac{1}{x}\right)^{-n} = \sum_{h=0}^{\infty} (-1)^h C_h^{n+h-1} x^{n-h}; \qquad |x| > 1. \tag{2.6}$$

Likewise the following expansion is of some importance:

$$\begin{aligned} x\frac{R^n + (-1)^n R}{R+1} + (-1)^{n-1}R = {} & \sum_{h=0}^{n-2} (-1)^h \left\{\sum_{t=0}^{h} C_t^{n-h-2+2t}\right\} x^{n-h} \\ & + \sum_{h=n-1}^{\infty} (-1)^h \left\{1 + \sum_{t=h+2-n}^{h} C_t^{n-h-2+2t}\right\} x^{n-h}. \end{aligned} \tag{2.7}$$

For the sake of completeness we add two further expansions to be used in the future:

$$(x-1)^n = \sum_{h=0}^{n} (-1)^h C_h^n x^{n-h}, \tag{2.8}$$

$$\begin{aligned} &(x-1)^n + x^{n-1} - (x-1)^{n-1} \\ &\qquad = x^n - nx^{n-1} + \sum_{h=2}^{n} (-1)^h \{C_h^n + C_{h-1}^{n-1}\} x^{n-h}. \end{aligned} \tag{2.9}$$

Let $f_n(x)$ and $g_n(x)$ be two functions developable in descending powers of x and beginning with the term x^n. Set

$$f_n(x) = x^n + \sum_{k=1}^{\infty} (-1)^k b_k x^{n-k}, \qquad g_n(x) = x^n + \sum_{h=1}^{\infty} (-1)^h c_h x^{n-h};$$

then (contrary to the more usual notation of the next chapter) we shall in this chapter write $f_n(x) \ll g_n(x)$ or $g_n(x) \gg f_n(x)$, if $b_h \leqq c_h$ for $h=1, 2, 3, \cdots$. Assuming $f_n^i(x) \ll g_n^i(x)$ for $i=1, 2$ and $n=1, 2, 3, \cdots$, we evidently have

$$f_n^1(x) + (-1)^s f_{n-s}^2(x) \ll g_n^1(x) + (-1)^s g_{n-s}^2(x), \qquad s = 1, 2, \cdots. \tag{2.10}$$

The relation is also transitive. That is, if $f_n(x) \ll g_n(x)$ and $g_n(x) \ll h_n(x)$, then $f_n(x) \ll h_n(x)$.

3. **The rigorous relation between the chromatic polynomials and the conjectured asymptotic formula.** The fundamental result of this section is the following theorem.

THEOREM. *For all maps P_{n+3}^k of the type specified above, we have*

$$Q_n^k(x) \ll U_n^k(x). \tag{3.1}$$

Proof. There are only 3 topologically distinct maps P_4^k and only one map P_3^k (cf. Birkhoff [4, p. 10]). It is therefore easy to verify the theorem for $n=0, 1$. Assuming inductively that the theorem is true for $n<m$ $(m \geqq 2)$, we shall prove that the theorem is true for $n=m$. That is, we shall prove that

$$Q_m^k \ll U_m^k. \tag{3.2}$$

Now, by Theorem II, §5, of the preceding chapter, we know that the above relation is true when $k=m+3$. We therefore make a second inductive hypothesis and assume that (3.2) holds when $k>l$ $(l \geqq 3)$, and our goal now is to prove that

$$Q_m^l \ll U_m^l. \tag{3.3}$$

It may happen that P_{m+3}^l has a region T which does not abut at V but which does abut at a vertex $A \in F$(9). Let the other two regions abutting at A be denoted by R and S (cf. fig. 17). R and S also abut at V. Shrinking the

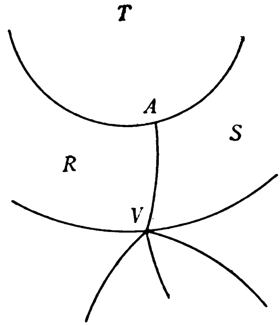


FIG. 17

boundary VA to a point, we get a map P^{l+1}_{m+3} for which (3.2) holds by our second inductive hypothesis. Also, without shrinking VA but by erasing it, we get a map P^{l-1}_{m+2} for which our first inductive hypothesis is valid, at least if P^{l-1}_{m+2} is actually a proper map. The special case, $l=3$, should here be parenthetically noted: In this case V is a vertex of P^{l-1}_{m+2} only by convention, which then has only triple vertices and is thus in reality a map P^3_{m+2} for which we have $Q^2_{m-1} = Q^3_{m-1} \ll U^3_{m-1} \ll U^2_{m-1}$. Returning to the general situation we evidently have:

Case 1. $Q^l_m = Q^{l+1}_m$ if P^{l-1}_{m+2} is a pseudo-map.

Case 2. $Q^l_m = Q^{l+1}_m - Q^{l-1}_{m-1}$ if P^{l-1}_{m+2} is a proper map.

In Case 1, $Q^l_m \ll U^{l+1}_m \ll U^{l+1}_m - U^{l-1}_{m-1} \equiv U^l_m$. In Case 2, we have $Q^l_m \ll U^{l+1}_m - U^{l-1}_{m-1} \equiv U^l_m$. In both cases we use (2.10) as well as the readily verifiable identity $U^l_m(x) \equiv U^{l+1}_m(x) - U^{l-1}_{m-1}(x)$.

Hence (3.3) has been proved if a region T of the type described above is available. But in the contrary case, we know by Theorem I, §5, of the preceding chapter that P^l_{m+3} has a two-sided region abutting at V. Erasing one of the sides of such a two-sided region, we obtain a map P^{l-1}_{m+2} to which our first inductive hypothesis may be applied. Furthermore $Q^l_m = (\lambda - 2) Q^{(l-1)}_{m-1} = x Q^{l-1}_{m-1} \ll x U^{l-1}_{m-1} \equiv U^l_m$. This completes the proof of the theorem.

The case of any map of simply connected regions and triple vertices is covered by (3.1) with $k=3$. From this formula we can now read off a rigorous relationship existing between the chromatic polynomial of such a map and the "asymptotic formula" of §1. It may be formulated as follows:

(9) As in the last section of the preceding chapter, $A \in F$ if, and only if, $A \neq V$ and there is a boundary line having A and V for its end points. The possibility of the second condition holding without the first is, of course, ruled out, as we are dealing with maps P^k_{n+3} of regions whose closures are simply connected.

$$\frac{P_n(\lambda)}{\lambda(\lambda-1)(\lambda-2)} \text{ is dominated for } \lambda < 2 \text{ by } \left[\frac{(\lambda-2)^2}{\lambda-1}\right]^{n-3}.$$

4. Lower bounds for the coefficients, $a_1, a_2, \cdots$. In the previous section we obtained results which yield upper bounds for the numbers $a_1, a_2, \cdots$. We now prove the following theorem which yields lower bounds and, in fact, is a refined form of the theorem proved by Birkhoff to the effect that the a's are all non-negative (cf. Birkhoff [4, p. 10]).

THEOREM. *If* P^k_{n+3} *(a map of the type considered above having one vertex* V *of multiplicity* k*) has the property that no pair of its regions have two or more sides in common*(10)*, then* $Q^k_n \gg x^{k-3}(x-1)^{n-k+3}$.

Proof. Exactly as in the previous proof, we may assume inductively that

$$Q^k_n \gg x^{k-3}(x-1)^{n-k+3} \tag{4.1}$$

firstly for $n < m$ and $3 \leqq k \leqq m+3$, and *secondly* for $n = m$ and $k > l$. We wish to prove that

$$Q^l_m \gg x^{l-3}(x-1)^{m-l+3}. \tag{4.2}$$

If P^l_{m+3} contains a proper three-ring, then by shrinking to a point the part of the map on the side of the ring which does not contain V we obtain a map of the type $P^l_{\alpha+3}$. Shrinking the side that does contain V we obtain a map of the type $P^3_{\beta+3}$, with $\alpha+\beta=m$, $0<\alpha$, $\beta<m$. Also we have $Q^l_m = Q^l_\alpha \cdot Q^3_\beta$ (cf. Chapter I (7.2)). Thus from (4.1) we have $Q^l_m \gg x^{l-3}(x-1)^{\alpha-l+3}\cdot(x-1)^\beta = x^{l-3}(x-1)^{m-l+3}$. Thus (4.2) must hold under the present circumstances. We may therefore restrict attention to the case when no proper three-ring exists in P^l_{m+3}.

It may happen that P^l_{m+3} has a region T which does not abut at V but which does abut at a vertex $A \in F$ (for definition of F, cf. §5, Chapter II). Let the other two regions abutting at A be denoted by R and S. Shrinking the side VA to the point V, we get a map P^{l+1}_{m+3} for which (4.1) holds by our second inductive hypothesis. Also, leaving the point A in its original position but obliterating the side VA, we get a map P^{l-1}_{m+2} for which our first inductive hypothesis is valid. For, in P^{l-1}_{m+2}, the region formed by the union of R and S can not be in a proper 2-ring, inasmuch as P^l_{m+3} has no proper 3-ring. Hence, if $l>3$, we have

$$\begin{aligned} Q^l_m &= Q^{l+1}_m - Q^{l-1}_{m-1} \gg x^{l-2}(x-1)^{m-l+2} - x^{l-4}(x-1)^{m-l+3} \\ &= x^{l-4}(x-1)^{m-l+2}[x^2 - x + 1] \gg x^{l-4}(x-1)^{m-l+2}(x^2 - x) \\ &= x^{l-3}(x-1)^{m-l+3}. \end{aligned}$$

---

(10) It is thus possible by our hypothesis for P^k_{n+3} to have two regions R and S such that $\overline{R}+\overline{S}$ is doubly connected, but the omission of the exceptional vertex V would render it simply connected. (Here $\overline{R}$ is the closure of R.)

But, if $l=3$, V is a vertex only by convention in P^{l-1}_{m+2}, and this map is a P^{3*2}_{m+}. Hence

$$Q^3_m = Q^4_m - Q^{3*}_{m-1} \gg x(x-1)^{m-1} - (x-1)^{m-1} = (x-1)^m.$$

Hence (4.2) has been proved if a region T of the type specified above is available. In the contrary case, we know from Theorem I, §5 of the preceding chapter, that P^l_{m+3} has a two-sided region abutting at V(11). Hence

$$Q^l_m = (\lambda-2)Q^{l-1}_{m-1} = xQ^{l-1}_{m-1} \gg x\cdot x^{l-4}(x-1)^{m-l+3} = x^{l-3}(x-1)^{m-l+3}.$$

This completes the proof.

It is easy to see how the above theorem can be modified so as to take care of the case when 2-rings are allowed. Namely, if the map has r proper two-rings, but otherwise satisfies the hypotheses of the theorem, we would have

$$Q^k_n \gg x^{r+k-3}(x-1)^{n-r-k+3}.$$

5. **Refinements of the results of the two preceding sections.** The theorems of the two previous sections can be somewhat sharpened, if we restrict attention to maps of sufficient regularity. Our results in this direction are given below in Theorems I and II. We must first, however, prove the following almost obvious lemma.

LEMMA. *If M_n is a proper map of triple vertices and n ($n>3$) simply connected regions, it contains a side which together with its end points is in contact with four distinct regions.*

Proof. The lemma is true for $n=4$, where there are only two topologically distinct possibilities (cf. Birkhoff [4, p. 11]). Assume inductively that the lemma is true for $4\leq n<m$. If now we assume the lemma false for $n=m$, we arrive at a contradiction as follows:

Consider a map M_m in which no side abutting four regions exists. Let R and S be two regions separated by a side AB whose end points are the vertices A and B(12). Let T be the third region abutting on AB. Then the point set E consisting of T and the side AB must be doubly connected. If either of the two collections of regions into which E separates its complement with respect to the sphere consisted solely of one region (R or S), the obliteration of the side AB would lead to a submap of $m-1$ regions of triple vertices for which the lemma would fail. It follows that T must have at least six vertices, since no multiple vertices are allowed and since no multiply connected regions are allowed. If the vertices of T are denoted in the order in which they occur by

(11) It should be noted that in this case we necessarily have $k>3$. Otherwise the map would contain a pair of regions having two sides in common, contrary to hypothesis.

(12) The existence of two vertices in any map of simply connected regions with triple vertices can never be in doubt. There are obviously at least $2m/3$ vertices.

$A_1, A_2, \cdots, A_6, \cdots$, it is clear by our assumption that the region T^1 in contact with T across A_1A_2 must also touch T across A_3A_4, thus making $T+T^1$ doubly connected. The third region abutting at A_3 must therefore be separated from the third region abutting at A_4. Hence A_3A_4 must be in contact with four distinct regions. This is the desired contradiction.

THEOREM I. *If P^3_{n+3} is a map of triple vertices with $n+3$ simply connected regions, then*

$$Q^3_n(x) \ll x\frac{R^n + (-1)^n R}{R+1} + (-1)^{n-1}R, \tag{5.1}$$

where the notation is explained in §2.

Proof. When $n=1$, this reduces to $Q^3_1(x) \ll R$, which is known to be true. We inductively assume the theorem true for $n<m$. Consider, then, a map P^3_{m+3}. By the lemma just proved, we know that it contains a side AB against which four distinct regions abut. If the two regions which have contact across AB have no further contact, the obliteration of AB leads to a map $P^3_{(m-1)+3}$ for which (5.1) is valid; otherwise we would get a pseudo-map. On the other hand, in either case, the shrinking of AB to a point leads to a map P^4_{m+3} with one quadruple vertex. We thus have either $Q^3_m = Q^4_m - Q^3_{m-1}$ or $Q^3_m = Q^4_m$. In either case, it follows from §3 and our inductive hypothesis that

$$\begin{aligned} Q^3_m(x) &\ll xR^{m-1} - x\frac{R^{m-1} + (-1)^{m-1}R}{R+1} - (-1)^{m-2}R \\ &= x\frac{R^m + (-1)^m R}{R+1} + (-1)^{m-1}R, \end{aligned}$$

and the theorem is proved.

THEOREM II. *If P^3_{n+3} is a map of triple vertices having $n+3$ simply connected regions with no proper 2-rings or 3-rings, then*

$$Q^3_n(x) \gg (x-1)^n + x^{n-1} - (x-1)^{n-1}. \tag{5.2}$$

Proof. According to Whitney there exists a simple closed curve C without multiple points passing through each region just once and crossing each side at most once, but not passing through any vertex (cf. Whitney [3]). The complement of C with respect to the sphere consists of two simply connected domains D_1 and D_2, separated by C(13). Each region of P^3_{n+3} contains vertices in both D_1 and D_2. Select any vertex V in D_1 and any region R_1 abutting V and by a continuous deformation of the map which leaves invariant the domain D_2 and is one-to-one except for the points which go into V_1 pull up

(13) This is the classic theorem of Jordan.

in succession all the vertices of R_1 lying in D_1 to coincide with V. At each step, say right after we have pulled in the $(k-3)$th vertex, we obviously have a map of the type P_{n+3}^k, with one k-tuple vertex, against which no region abuts more than once. For, if a region abutted V more than once, either it would exclude at least one region from being entered by C or it would itself be entered at least twice by the curve C, contrary to the construction of C. Furthermore, with the deletion of V, the union of no three regions can be multiply connected in P_{n+3}^k.

Still using the notation of §2, we have

$$Q_n^k(x) = Q_m^{k+1}(x) - Q_{n-1}^{k-1}(x), \qquad k = n+3, \cdots, 4, \tag{5.3}$$

where the map P_{n+2}^{k-1} referred to by $Q_{n-1}^{k-1}(x)$ contains no pair of regions having two sides in contact, inasmuch as P_{n+3}^k has no proper ring of three regions, as well as no ring of two regions.

We shall prove first by induction on k that

$$Q_n^k \gg x^{k-4}(x-1)^{n-k+4} + x^{n-1} \qquad \text{for } k = n+3, n+2, \cdots, 5, 4. \tag{5.4}$$

For $k=n+3$, this gives $Q_n^{k+3} \gg x^n$, which is known to be true (cf. Theorem II, §5, Chapter II). Assume inductively that (5.4) holds for $k > l \geqq 4$. We shall prove (5.4) for $k=l$. By (4.1), we know that $Q_{n-1}^{k-1} \gg x^{k-4}(x-1)^{n-k+3}$. Combining this with (5.3) and our inductive hypothesis, we find that

$$Q_m^l \gg x^{l-3}(x-1)^{n-l+3} + x^{n-1} - x^{l-4}(x-1)^{n-l+3} = x^{l-4}(x-1)^{n-l+4} + x^{n-1}.$$

This proves (5.4) for $k=4, 5, \cdots, n+3$. For $k=3$, we have, however, $Q_n^3(x) = Q_n^4(x) - Q_{n-1}^{3*}(x)$, since V would be a vertex by convention only for P_{n+3}^2. Hence, using the fact that $Q_{n-1}^{3*} \gg (x-1)^{n-1}$ by (4.1) and also the fact just proved for $k=4, \cdots$, we find that (5.2) holds, as desired.

6. **Recapitulation of the inequalities proved in §§3, 4, 5, in the case of maps with triple vertices only.** Taking $k=3$ and comparing (3.1) and (4.1) with (2.6) and (2.8), we obtain the very interesting inequalities,

$$C_h^n \leqq a_h \leqq C_h^{n+h-1}, \qquad h = 1, 2, 3, \cdots, n, \tag{6.1}$$

Likewise, comparing (5.1) and (5.2) with (2.7) and (2.9), we obtain the still sharper inequalities,

$$C_h^n + C_{h-1}^{n-1} \leqq a_h \leqq \sum_{t=0}^{h} C_t^{n-h-2+2t}, \qquad h = 2, 3, \cdots, n-2 \tag{6.2}$$

The conditions under which these various inequalities hold are not the same and are specified in detail in the preceding paragraphs. We merely note here that they all hold for regular maps.

It is to be noted that (6.1) determines the exact value of a_1, while a_2 is determined exactly by (6.2), with the result that for regular maps of $n+3$ regions, we must have

$$(6.3)\qquad a_1 = n, \qquad a_2 = (n+2)(n-1)/2.$$

These values for a_1 and a_2 can, however, be determined by any one of a number of other methods (cf. Birkhoff [4, p. 19]). The complete set of inequalities (6.1) or (6.2) seems to constitute a much deeper result.

7. **A determinant formula for a chromatic polynomial developed in powers of $\lambda-2$.** Birkhoff has given a general explicit formula for the chromatic polynomials in terms of the number of ways the maps can be broken down into submaps in various numbers of steps (cf. Birkhoff [1]; also Whitney [1]). In this formula the polynomial is developed in powers of λ. We now give a similar formula (cf. (7.2) below) which yields us the polynomial developed in powers of $\lambda-2$.

The main features of the proof of our new formula are given in this section. Several lemmas which are needed to make this proof rigorous are deferred to §9. In §8, we give a simple example of the application of the new formula.

Except where otherwise stated, we restrict attention to maps of at least three regions, all of whose vertices are triple and none of whose regions are multiply connected.

Let us "mark" some of the boundaries of a map of n regions in such a manner that it is possible to number the regions of the map $1, 2, \cdots, n$ in such wise that region 2 is in contact with region 1 across an unmarked boundary; region 3 is in contact with regions 1 and 2 across unmarked boundaries; region 4 is in contact with just two of the regions 1, 2, 3 across unmarked boundaries; and, in general, region k $(2<k\leqq n)$ is in contact with just two of the regions $1, 2, \cdots, k-1$ across unmarked boundaries, *these two regions themselves being in contact with each other across unmarked boundaries.* The possibility of such markings will be discussed later.

If the boundaries of a map are marked so as to fulfill these conditions, we shall, for brevity, speak of the map itself as having been marked. Also, if it is possible to number the regions as described above without marking *any* of the boundaries, we shall, in certain circumstances, think of the map as having been marked, the set of the marked boundaries being vacuous.

Assuming that a map has been marked, it is obvious that, if we neglect whether or not the colors match across the *marked* boundaries, the number of ways the map can be colored in λ colors is $\lambda(\lambda-1)(\lambda-2)^{n-2}$. For, coloring the regions in the order in which they may be numbered so as to fulfill the above requirements, we see that any one of the λ colors may be assigned to the region 1; any one of the $\lambda-1$ colors different from the color assigned to region 1 may be assigned to region 2; any one of the $\lambda-2$ colors different from the two colors assigned to regions 1 and 2 may be assigned to region 3; and, in general, any one of the $\lambda-2$ colors different from the two colors (necessarily distinct) already assigned to the two of the regions $1, 2, \cdots, k-1$, with which region k is in contact across unmarked boundaries, may be assigned to region k. Such a coloration, in which it is not required that two regions in contact across a marked boundary should have different colors, will be called for brevity a coloring of the marked map.

It is *not* possible in all cases to mark a map in the sense just explained. A simple example to show this is indicated in figure 18. Nevertheless, in figure 19 is indicated a map chromatically equivalent to the map of figure 18 which

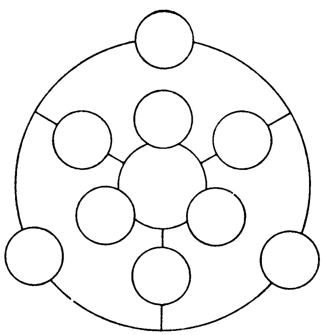


FIG. 18

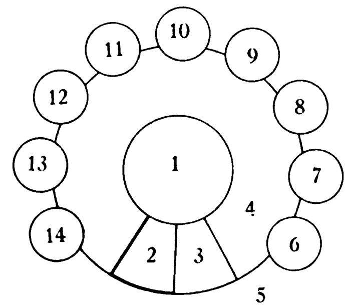


FIG. 19

can be marked[14]. This situation is general. According to Lemma 3, §9, there is in any complete set of chromatically equivalent maps at least one which can be marked, at least in a certain slightly generalized sense which does not affect present considerations.

Let P be a map which can be marked. We consider some one particular fixed marking. Then, any coloring of the marked map is an ordinary coloring of some submap of P in which certain of the marked boundaries have been erased but all the unmarked boundaries have been left inviolate. Namely the marked boundaries to be erased are just those boundaries which separate similarly colored regions. Conversely, any ordinary coloring of any submap of P obtained by erasing marked boundaries only is a coloring of the marked map P. Hence, if we denote by $P^1=P$[15], $P^2, P^3, \cdots, P^k$ all absolutely distinct proper submaps of P obtained by erasing marked boundaries only, we see that $\sum_{i=1}^{k}P^i(\lambda)$ is equal to the total number of ways of coloring the marked map P. Thus we have

$$\lambda(\lambda-1)(\lambda-2)^{n-2}=\sum_{i=1}^{k}P^i(\lambda), \tag{7.1}$$

where n is the number of regions in P.

[14] The chromatic polynomial for both maps is easily seen to be $\lambda(\lambda-1)(\lambda-2)^{10}(\lambda-3)^2$.

[15] Every map is considered to be a submap of itself in the present context.

It will be proved later (Lemma 3, §9) that none of the submaps $P^2, P^3, \cdots, P^k$ has any multiply connected regions, the same being assumed at the outset to be true of the map $P=P^1$. We next replace, whenever necessary, the maps $P^2, P^3, \cdots, P^k$ by chromatically equivalent maps which can be marked. Since no confusion can result, we denote any map chromatically equivalent to P^i also by P^i, and when we speak briefly of marking the map P^i we are really contemplating the double operation of (firstly) choosing a particular map out of the appropriate class of chromatically equivalent maps and then (secondly) of marking the chosen map. With this understanding we can repeat the above process with each of the submaps $P^2, P^3, \cdots, P^k$. The submaps obtained from P^2 by erasing only marked boundaries of some particular marking of P^2 are called $P^2, P^{k+1}, P^{k+2}, \cdots, P^{k+r}$. The submaps of P^3 are called $P^3, P^{k+r+1}, P^{k+r+2}, \cdots, P^{k+r+s}$, and so forth.

Our future use of the word "step" is made sufficiently precise by observing that P^{k+1} is derived from P^2 in one step and from the original map P in two steps, regardless of the number of marked boundaries that were erased in passing from P to P^2 or from P^2 to P^{k+1}, this number being in each case not less than 1. However, a map, when considered as a submap of itself, is regarded as derived from itself in zero steps.

It is to be understood that all the maps obtained in two steps are treated in the same way to give rise to still more maps obtained at the third step; and the process is repeated until after a finite number of steps all the marked maps have vacuous sets of marked boundaries. Here the process comes to a natural halt after we have obtained a total of, say, m maps (cf. Lemma 4, §9).

Of course, the number of ways these m maps can be obtained is enormous. For it is very arbitrary as to just how a map may be marked at each step. We emphasize that we are now considering only some one fixed choice. It should also be remembered that these maps $P^1, P^2, \cdots, P^m$ are not necessarily distinct from each other.

Next we let the symbol $[i, j]$ represent the number of these maps $P^1, P^2, \cdots, P^m$ which contain just j regions and are derived from P in just i steps[16]. We shall now prove the formula

$$P(\lambda) = \lambda(\lambda - 1)\left[\sum_{i,j} (-1)^i [i, j](\lambda - 2)^{j-2} \right]. \tag{7.2}$$

Proof. For each of the P^i we evidently have an equation like (7.1), which we write in the form

$$\lambda(\lambda - 1)(\lambda - 2)^{n_i-2} = \sum_{j=1}^{m} E_{ij}P^j(\lambda), \qquad i = 1, 2, \cdots, m, \tag{7.3}$$

(16) The following obvious equalities satisfied by these bracket symbols are of some interest: $[0, n]=1$; $[0, j]=0$, if $j<n$; $[i, j]=0$, if $i+j>n$; $\sum_{i,j}[i, j]=m$, the total number of maps in the sequence $P^1, P^2, \cdots, P^m$.

where n_i is the number of regions in the map P^i and where $E_{ij}=1$ if the map P^j is obtained from P^i in one step or in zero steps (in the case of E_{ii}) but is otherwise zero. According to the notation chosen P^j can be derived from P^i only if $j\geqq i$, that is, $E_{ij}=0$ if $i>j$, while $E_{ii}=1$ for all i. Hence the determinant $|E_{ij}|$ is equal to unity and we can solve the m equations (7.3) for $P^1(\lambda)=P(\lambda)$ in the form

$$P(\lambda) = \lambda(\lambda - 1)\begin{vmatrix} (\lambda-2)^{n-2} & E_{12} & E_{13} & E_{14}\cdots E_{1m} \\ (\lambda-2)^{n_2-2} & E_{22} & E_{23} & E_{24}\cdots E_{2m} \\ (\lambda-2)^{n_3-2} & 0 & E_{33} & E_{34}\cdots E_{3m} \\ (\lambda-2)^{n_4-2} & 0 & 0 & E_{44}\cdots E_{4m} \\ \vdots & \vdots & \vdots & \vdots \\ (\lambda-2)^{n_m-2} & 0 & 0 & 0\cdots E_{mm} \end{vmatrix}.$$

The value of the determinant is equal to the sum of all terms of the type

$$(-1)^{\iota}E_{1\alpha}E_{2\beta}\cdots E_{l-1,\gamma}(\lambda-2)^{n_l-2}E_{l+1,\delta}\cdots E_{m\epsilon},$$

where ι is the number of inversions in the permutation $[\alpha, \beta, \cdots, \gamma, 1, \delta, \cdots, \epsilon]$ of the m numbers $1, 2, 3, \cdots, m$. Now suppose that this permutation is factored into a product of cycles. The factors in the typical term can be correspondingly rearranged so that we may write it in the form

$$(-1)^{\iota}[(\lambda-2)^{n_l-2}E_{1a}E_{ab}E_{bc}\cdots E_{dl}][E_{pq}E_{qr}\cdots E_{sp}]\cdots[E_{xx}][E_{yy}]\cdots,$$

where $a, b, c, d, l, p, q, r, s, x, y$, and so on, are mutually distinct and where the brackets correspond respectively to the cycles $(l\ 1\ a\ b\ c\cdots d)$, $(p\ q\ r\cdots s), \cdots, (x), (y), \cdots$. Here, without loss of generality, we take as the first bracket the one which includes the element from the first column, namely $(\lambda-2)^{n_l-2}$. If N_t+1 is the number of elements included in the tth bracket, then $\iota\equiv\sum_t N_t \pmod 2$ and we may therefore replace ι by $i=\sum_t N_t$.

Since $E_{\alpha\beta}$ vanishes when $\alpha>\beta$, it is clear that, for nonvanishing terms,

$$\begin{gathered} 1<a<b<c<\cdots<d<l, \\ p<q<r<\cdots<s<p, \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{gathered} \tag{7.4}$$

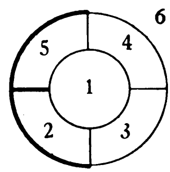
But, since it is absurd for p to be less than itself, as (7.4) would imply, and since $E_{xx}=E_{yy}=\cdots=1$, it is clear that all nonvanishing terms are of the type

$$(-1)^{i}E_{1a}E_{ab}E_{bc}\cdots E_{dl}(\lambda-2)^{n_l-2},$$

where $i=N_1$ is the number of the factors $E_{1a}\cdots E_{dl}$. By definition of the E's, a term of this type is equal to $(-1)^i(\lambda-2)^{n_l-2}$ or to zero according as to whether the map P^l is derived from P in i steps through the intermediate maps $P^a, P^b, P^c, \cdots, P^d$, or whether it is not so derived. Each of the maps

$P^1, P^2, \cdots, P^m$ is thought of as derived from P in just one way. Hence the sum of all terms of this type, for which i is fixed but l ranges through the values for which n_l, the number of regions in P^l, is equal to a fixed number j, is evidently $(-1)^i[i,j](\lambda-2)^{j-2}$. Summing over both i and j we get the value of the determinant and the desired formula (7.2) has been established.

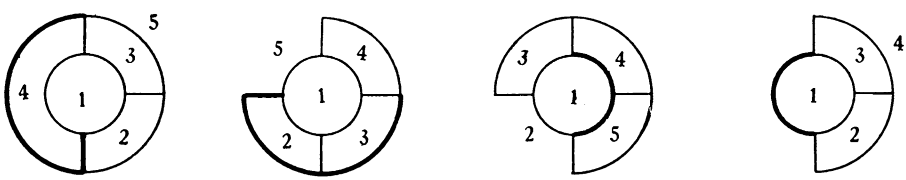
8. **Illustration of the determinant formula.** Let P be the map illustrated in figure 20. We mark it as there shown, indicating the marked boundaries



P^1 with 6 regions

FIG. 20

by heavier lines. Then we evidently get a total of four maps at the first step. These maps are shown in figure 21. These are also arbitrarily marked as



P^2 with 5 regions. P^3 with 5 regions P^4 with 5 regions P^5 with 4 regions

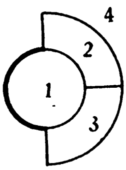
FIG. 21. Maps obtained at first step.

shown. At the second step we get the seven maps shown in figure 22. After marking these, we finally get at the third step the three maps of figure 23. By counting these maps we find $[0, 6]=1$, $[1, 5]=3$, $[1, 4]=1$, $[2, 4]=6$, $[2, 3]=1$, $[3, 3]=3$, with all other bracket symbols vanishing. This gives

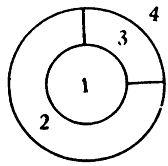
$$\begin{aligned} P(\lambda) &= \lambda(\lambda-1)\{[0,6](\lambda-2)^4 - [1,5](\lambda-2)^3 - [1,4](\lambda-2)^2 \\ &\qquad + [2,4](\lambda-2)^2 + [2,3](\lambda-2)^1 - [3,3](\lambda-2)^1\} \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-2)^3 - 3(\lambda-2)^2 + 5(\lambda-2) - 2]. \end{aligned}$$

This agrees with the Q polynomial listed in §2, Chapter II, for the map (6; 6). For computing chromatic polynomials the present method is of little use, as it is far more complicated than the method indicated in the first section of Chapter II.

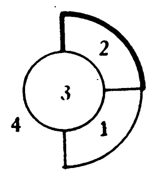
The determinant formula will probably also prove to be of little use for theoretical purposes. It is hard to believe, for instance, that it will ever yield more precise results than the inequalities of §6. In fact, the known properties of the chromatic polynomials yield a great deal more information about the $[i, j]$ symbols than the $[i, j]$ symbols yield about the chromatic polynomials.



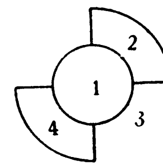
P^6 with 4 regions



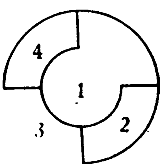
P^7 with 4 regions



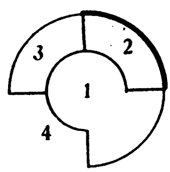
P^8 with 4 regions



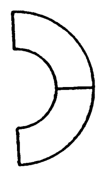
P^9 with 4 regions



P^{10} with 4 regions

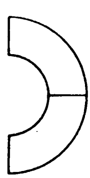


P^{11} with 4 regions

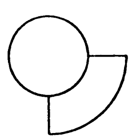


P^{12} with three regions

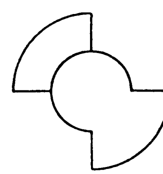
FIG. 22. Maps obtained at second step.



P^{13} with 3 regions



P^{14} with 3 regions



P^{15} with 3 regions

FIG. 23. Maps obtained at third step.

9. **Proofs of the deferred lemmas.** The following lemmas, needed for the rigorous discussion of the determinant formula, were postponed to the present section so as not to interrupt the continuity of exposition in §7.

LEMMA 1. *In any map of n simply-connected regions having all of its vertices triple save for one n-tuple vertex against which each region abuts just once, there exists at least one two-sided region* ($n \geqq 3$).

This lemma has already been proved. It is a special case of Theorem I, §5, Chapter II.

LEMMA 2. *It is possible to mark any map of triple vertices and n simply-connected regions which contains no proper 2- or 3-rings. Furthermore, this can be done in such a way that the three boundaries which abut on a preassigned vertex are unmarked, and so that the number of marked boundaries is $n-3$, while the number of boundaries having no vertices in common with the unmarked boundaries is also $n-3$.*

Proof. Draw a Whitney curve, that is, a closed curve without multiple points passing through each region once and only once but passing through no vertex. This curve divides the map into two parts. Let the part which contains the assigned vertex be called the outside, while the other part is called the inside of the Whitney curve. If we deform the map continuously, so that the inside of the curve is shrunk to a point while the outside is transformed in a one-to-one continuous manner onto the rest of the sphere, it is clear that we get a map of the type referred to in Lemma 1. But such a map can be colored in $\lambda(\lambda-1)(\lambda-2)^{n-2}$ ways (cf. Theorem II, §5, Chapter II). Hence the original map can be colored also in $\lambda(\lambda-1)(\lambda-2)^{n-2}$ ways *provided* that collisions of color which occur inside the Whitney curve be neglected. Hence we mark all the boundary lines which lie *entirely* inside the Whitney curve. We leave it to the reader to show in detail how the regions may be numbered from 1 to n in the manner described in §7. Suffice it to say that the number n is assigned to the region which corresponds to the 2-sided region mentioned in Lemma 1. After erasing one of the two sides of this 2-sided region, we get a map of the same type with one less region. This map also has a 2-sided region and its position indicates the region in the original map to which we assign the number $n-1$. In this way all the regions are numbered in reverse order.

To prove the last part of the lemma, we note that, if we erase the boundary lines and the partial boundary lines which lie outside the Whitney curve, we obtain a map of $n+1$ regions, provided we use the Whitney curve itself as a set of additional boundaries. It is well known from Euler's polyhedral formula that such a map has $3(n+1)-6=3n-3$ boundary lines (cf. Birkhoff [4, p. 3]). Of these, n are the boundary lines lying along the Whitney curve; n more are boundary lines abutting on the Whitney curve, while the remaining $n-3$ boundary lines are entirely inside the Whitney curve. These are just the ones which we marked. The fact that the number of boundaries not having a vertex in common with the marked boundaries is also $n-3$ follows by symmetry with respect to the Whitney curve, as these are precisely the boundaries on the outside thereof.

COROLLARY. *The number of unmarked vertices (that is, vertices which are not the end points of any marked boundary lines) is $n-2$, if $n>3$. In case $n=3$, the number is 2.*

In fact, for $n>3$, the boundaries which have no vertex in common with the marked boundaries, together with their end points, form a tree, since they form a simply connected set. Otherwise the Whitney curve would not pass through every region. The relation between the number of vertices and the number of edges of a tree is well known. The corollary then follows from the last clause of Lemma 2.

The case $n=3$ is an exception because the set of marked boundaries in this case alone is vacuous.

Before stating Lemma 3, we describe two operations for passing from a given map to another chromatically equivalent map.

The first one may be used in case the map contains a proper 2-ring. Any pair of regions R_1 and R_2, whose union is multiply connected with multiplicity r, divides the map M into a number of isolated configurations $F_1, F_2, \cdots, F_r$ $(r \geqq 2)$, such that $F_1+F_2+ \cdots +F_r+R_1+R_2=M$, the whole map. Select one of these configurations, say F_1; deform the map so that F_1 fits into the interior of a circle C^1, which is then thought of as removed from the map by a rigid motion in space, leaving the boundary of the circle behind on the map. This circular boundary is now shrunk to a point Q. Next select any point P in the interior of any boundary line of the map([17]) as so far modified. Expand P into a circular 2-sided region C of the same radius as that of C^1, while the rest of the map suffers a one-to-one continuous transformation. By a rigid motion the configuration F_1 is returned to the map in a new position, namely into the circle C. The map as thus modified will be chromatically equivalent to the original map, at least if we take care by a further deformation of F_1 (if necessary) that the vertices on the circular boundary of F_1 are in the same cyclic order after the modification as the corresponding vertices were in the original map. This follows from (7.1), Chapter I.

The second operation may be used in case the map contains a 3-ring. It is similarly described. The only modification is that the point P is taken at a vertex instead of at an interior point of a boundary line. The point P is expanded into a circular three-sided region into which the configuration F_1 is placed.

Suppose that we fix attention upon a certain set S of boundaries of a map. Let S^1 be the complementary set. Suppose further that $P(\lambda)$ denotes the number of ways the map may be colored in λ colors so that no two regions which have contact across boundaries in the set S^1 shall have the same color, while collisions of color across boundaries belonging to S are completely neglected. Then the reduction formulas for 2- and 3-rings (that is, (7.1) and (7.2) of Chapter I) still hold for these modified chromatic polynomials, provided that the boundaries separating the regions in the ring do not belong to S (that is, they do belong to S^1). This obvious principle is fundamental in the proof of Lemma 3, where the set S is the set of marked boundaries.

([17]) Or, more generally, of any chromatically equivalent map. This generalization is actually needed for the proof of Lemma 3.

LEMMA 3. *Any map of simply connected regions and triple vertices containing perhaps two-rings or/and three-rings is chromatically equivalent to a map which can be marked, in such wise that at least one vertex is unmarked, provided at least that we accept the following slight generalization of the term "marked": A map of n regions is marked if the number of ways it can be colored in* λ *colors in such wise that no two regions with an unmarked boundary in common shall have the same color (but without regard to the matching of colors at the marked boundaries) is* $\lambda(\lambda-1)(\lambda-2)^{n-2}$[18].

Proof. The proof is by induction on the number n of the regions in the map. It is readily verified that the theorem is true for $n=3$, when the set of markings is vacuous and both vertices are unmarked.

The case when no 2- or 3-rings are present is disposed of in Lemma 2. In the case to be considered, then, we can evidently find a proper 2-ring or 3-ring R which has the property that, if one of the two isolated configurations F_1 or F_2 (say F_2) into which R divides the map is shrunk to a point, the resulting map has no proper 2- or 3-rings. If we shrink F_i ($i=1$ or 2) to a point we obtain a map M_i of fewer regions than the original map M. Hence, by our inductive hypothesis, M_i can be marked so that there is at least one unmarked vertex; or at least a map M_i^* chromatically equivalent to M_i can be so marked. However, we must take care, when M_1^* and M_2^* are combined to form a map M^* chromatically equivalent to M, that none of the marked boundaries occur between the regions of R. As a matter of fact, this is not always possible. In virtue, however, of the fact that M_2 has no proper 2- or 3-rings, we may by Lemma 2 choose the markings so that the crucial boundaries[19] are not marked in the map $M_2=M_2^*$. The situation with respect to M_1 or M_1^* is not so fortunate, but M_1^* has at least one unmarked vertex by our inductive hypothesis, so that F_1 can be replaced elsewhere according to one of the two operations described above for passing to a chromatically

[18] This definition differs from the former merely in that we are not now required to enumerate the regions in the manner prescribed in §7. Probably this enumeration can always be carried out for maps that can be marked in this generalized sense, but, as this seems a bit difficult to prove and as the enumeration is actually not needed, it seemed well to introduce the above generalization. It is odd that the method which is easiest in its application to simple examples is the theoretically more difficult. This last remark applies also to regular maps, where perhaps the easiest way to draw a Whitney curve is to carry out the enumeration first (more or less at random) and mark the boundaries as we go along so as to suppress unwanted contacts. No matter how this is done, in most cases the enumeration can be completed and the set S of marked boundaries will turn out to be simply connected and will touch each region just once. Hence a closed curve drawn completely around S and in the neighborhood of S will be a Whitney curve. Yet for theoretical purposes, in connection with Lemma 2, we reverse the process by drawing the Whitney curve first, marking the map secondly, and enumerating the regions (in reverse order) last.

[19] There is only one such boundary if R is a 2-ring and three such having a common vertex if R is a 3-ring.

equivalent map. When M^* has been thus formed, it will be seen that the markings of the constituent parts M_1^* and M_2^* yield a marking for the whole, at least in the slightly generalized sense specified above.

Furthermore, by the corollary to Lemma 2, we may calculate the number of unmarked vertices in M_2 in terms of f, the number of regions in F_1. In case R is a two-ring, M_2 has $f+2$ regions and hence at least f unmarked vertices, all of which occur in (or on the boundary of) the configuration F_1. In case R is a 3-ring, M_2 has $f+3$ regions and hence $f+1$ unmarked vertices, f of which occur in (or on the boundary of) the configuration F_1. In either case, as $f \geqq 1$, the map M^*, chromatically equivalent to M and marked in the manner described, will have at least one unmarked vertex and our proof by induction is complete.

LEMMA 4. *Any proper submap of a map marked in accordance with the directions given in Lemmas 2 and 3, which is obtained by erasing marked boundaries only, can not contain multiply connected regions.*

Proof. First we prove the lemma for the case of a map with no proper 2- or 3-rings, where all the marked boundaries are on the inside of a Whitney curve. In order to obtain a multiply connected region in the submap, it must be possible to find a set of regions $R_1, R_2, \cdots, R_k$, such that R_i is in contact with R_{i+1} ($i, i+1$ taken modulo k) across a marked boundary and such that $\sum_{i=1}^k \overline{R}_i$ is multiply connected. Since all the marked boundaries are inside the Whitney curve and since this latter enters each region just once, it will be possible to draw a simple closed curve Γ through $R_1, R_2, \cdots, R_k$, passing just once through each R_i in the order named but not entering any other region *and lying entirely inside the Whitney curve.* Furthermore, Γ must have at least one region completely on each side of itself (since $\sum \overline{R}_i$ is doubly connected). This would imply the existence of a region completely inside the Whitney curve, contrary to the definition of the latter.

The proof for maps containing proper 2-rings and/or proper 3-rings can be carried through by induction. If we shrink one side of such a ring to a point we get a map of fewer regions. Our inductive hypothesis is therefore to the effect that a multiply connected region in the submap can not be obtained from regions in the original map lying entirely on one side of any proper 2-ring or 3-ring. But the existence of any other such multiply connected region in the submap would be absurd as our configuration of boundaries for the submap would then necessarily contain an isthmus. We omit details.

LEMMA 5. *Any submap of the type described in Lemma 4 has at least three regions.*

Proof. The lemma is obvious for maps without 2- or 3-rings, as the number of marked boundaries for a map of n regions is only $n-3$, and every time we erase a boundary we decrease the number of regions by just one and decrease the number of marked segments (originally marked boundaries) by at least one.

The remaining cases are easily reduced to the case already treated.

CHAPTER IV. EXPANSIONS IN POWERS OF $\lambda-4$ AND $\lambda-5$

1. **Notation.** We are concerned throughout this chapter exclusively with proper maps of simply connected regions all of whose vertices are triple. In connection with such a map P_{n+3} of $n+3$ regions, we use the following notation:

$$x = \lambda - 4; \qquad y = \lambda - 5, \tag{1.1}$$

$$Q_n(\lambda) = R_n(x) = S_n(y) = \frac{P_{n+3}(\lambda)}{\lambda(\lambda-1)(\lambda-2)}. \tag{1.2}$$

Let $f(\xi)$ and $g(\xi)$ be two polynomials in ξ. Then (contrary to the notation of the preceding chapter) we shall in this chapter write $f(\xi) \ll g(\xi)$ or $g(\xi) \gg f(\xi)$ if, and only if, the coefficients of $f(\xi)$ are non-negative and not greater than the corresponding coefficients of $g(\xi)$.

We also write

$$T(\lambda) \ll U(\lambda) \qquad \text{for } \lambda \geqq c, \tag{1.3}$$

when T and U are polynomials in λ and c is a constant, if, upon setting $\xi=\lambda-c$, $f(\xi)=T(\lambda)$, and $g(\xi)=U(\lambda)$, we have $f(\xi) \ll g(\xi)$ in accordance with the preceding definition. It is evident that, if (1.3) holds, then $T(\lambda) \ll U(\lambda)$ for $\lambda \geqq b$, if $b > c$, so that our definition is consistent with the ordinary meaning of the symbol $\geqq$.

2. **Powers of $\lambda-4$.** The following theorem is fundamental:

THEOREM I. *If P_{n+3} has a proper 2-ring or a proper 3-ring or a four-sided region K surrounded by a proper 4-ring, then*

$$(\lambda-3)^n \ll Q_n(\lambda) \ll (\lambda-2)^n \qquad \text{for } \lambda \geqq 4, \tag{2.1}$$

provided that this same relation, with n replaced by m, holds for certain maps P_{m+3} with $m<n$.

The detailed specification of these "certain maps" is given implicitly in the following proof. This specification is not important. The one important feature is that each of these maps has fewer regions than the given map P_{n+3}. This theorem does not lend itself (except under very limited conditions, in the proof of Theorem II below, for instance) to complete induction, since the hypothesis concerning P_{n+3} to the effect that it has a 2- or 3-ring or a four-sided region need not hold for all the maps P_{m+3}.

Proof. Formulas (7.1), (7.2), and (2.5), of Chapter I, may be written in our present notation as follows:

$$(2.2)\qquad R_n(x) = (x+2)R_\alpha(x)R_\beta(x), \qquad \alpha+\beta+1 = n.$$

$$(2.3)\qquad R_n(x) = \overline{R}_\alpha(x)\overline{R}_\beta(x), \qquad \alpha+\beta = n,$$

$$(2.4)\qquad R_n(x) = \frac{x}{2}\left[R_{n-1}^{(1)}(x) + R_{n-1}^{(2)}(x)\right] + \frac{x+2}{2}\left[R_{n-2}^{(1)}(x) + R_{n-2}^{(2)}(x)\right].$$

Assuming that

$$(2.5)\qquad (x+1)^\gamma \ll R_\gamma(x) \ll (x+2)^\gamma, \qquad \gamma = \alpha \text{ and } \beta,$$

$$(2.6)\qquad (x+1)^\gamma \ll \overline{R}_\gamma(x) \ll (x+2)^\gamma, \qquad \gamma = \alpha \text{ and } \beta,$$

$$(2.7)\qquad \begin{aligned} (x+1)^{n-1} \ll R_{n-1}^{(i)}(x) \ll (x+2)^{n-1}, \qquad i = 1 \text{ and } 2,\\ (x+1)^{n-2} \ll R_{n-2}^{(i)}(x) \ll (x+2)^{n-2}, \qquad i = 1 \text{ and } 2,\end{aligned}$$

we are to prove that

$$(2.8)\qquad (x+1)^n \ll R_n(x) \ll (x+2)^n.$$

If P_{n+3} has a proper two-ring, (2.2) and (2.5) may be used to prove (2.8) in the following way: From (2.5) we have

$$(x+1)^{\alpha+\beta} \ll R_\alpha(x)R_\beta(x) \ll (x+2)^{\alpha+\beta}.$$

Hence, multiplying by $(x+2)$ and using (2.2), we get

$$(x+1)^{\alpha+\beta+1} \ll (x+1)^{\alpha+\beta}(x+2) \ll R_n(x) \ll (x+2)^{\alpha+\beta+1}.$$

Since in this case $\alpha+\beta+1=n$, (2.8) has been proved.

If P_{n+3} has a proper three-ring, (2.3) and (2.6) are similarly applicable.

If P_{n+3} has a four-sided region K surrounded by a proper four-ring, we use (2.4) and (2.7) as follows: From (2.7) we have

$$x(x+1)^{n-1} \ll \frac{x}{2}\left[R_{n-1}^{(1)}(x) + R_{n-1}^{(2)}(x)\right] \ll x(x+2)^{n-1}$$

and

$$(x+2)(x+1)^{n-2} \ll \frac{x+2}{2}\left[R_{n-2}^{(1)}(x) + R_{n-2}^{(2)}(x)\right] \ll (x+2)^{n-1}.$$

Adding and making use of (2.4), we obtain

$$(x^2+2x+2)(x+1)^{n-2} \ll R_n(x) \ll (x+1)(x+2)^{n-1}.$$

The proof is completed by noting that $(x+1)^n \ll (x^2+2x+2)(x+1)^{n-2}$ and $(x+1)(x+2)^{n-1} \ll (x+2)^n$.

THEOREM II. *The relations* (2.1) *always hold for* $n=0, 1, 2, \cdots, 8$.

Proof. For $n=0$, (2.5) holds, inasmuch as necessarily $R_0(x)=1$. Likewise for $n=1$, we have $R_1(x)=x+1$ or $x+2$, so that the theorem certainly holds in this case also. The theorem then follows by induction from the previous theorem and the fact that Euler's polyhedral formula shows the existence, in every map of triple vertices with less than twelve regions, of at least one region of two, three, or four sides. For such maps of $n+3$ regions, it is easily seen that the hypothesis of Theorem I must be fulfilled for $1<n<9$.

It is our conjecture that (2.1) *always* holds for maps of triple vertices. Every attempt to prove this has broken down on account of the possibility of the occurrence of maps without proper two-rings, three-rings, or four-sided regions. Probably the only maps of this type with fewer than seventeen regions are the maps listed in Chapter II with the symbols (12; 0, 12), (14; 0, 12, 2), (15; 0, 12, 3), (16; 0, 12, 4) (two maps), and (16; 0, 14, 0, 2), or maps topologically equivalent to these. It has been verified experimentally that (2.1) holds for each of these six maps as well as for the map (17; 0, 12, 5). In accordance with Theorem I, this almost certainly establishes the truth of our conjecture (2.1) for $n<14$; that is, for all maps having fewer than seventeen regions.

Needless to say our conjecture is a very strong form of the four-color conjecture. For, if (2.1) were true of all maps (of triple vertices), then, in particular, we could write $1 \leqq Q_n(4)$ for any map P_{n+3}. Hence P_{n+3} could be colored in four colors.

3. **Powers of $\lambda-5$.** The situation, which is very obscure for $\lambda \geqq 4$, is very clear for $\lambda \geqq 5$. We summarize the facts for this latter case in the following theorem, part of which has already been published (cf. Birkhoff [3 and 4]).

THEOREM. *For all maps of triple vertices*

$$(\lambda-3)^n \ll Q_n(\lambda) \ll (\lambda-2)^n \qquad \textit{for } \lambda \geqq 5. \tag{3.1}$$

Proof. This is already known to be true for $n<9$ by Theorem II of the preceding section. To prove the theorem for $n \geqq 9$, we assume inductively that (3.1) is true when $n<m$, and show that it must therefore hold for $n=m$. To this end we write formulas (7.1), (7.2), (2.5), and (3.4) of Chapter I in terms of y and S (cf. §1) as follows:

$$S_m(y) = (y+3)S_\alpha(y)S_\beta(y), \qquad \alpha+\beta+1=m, \tag{3.2}$$

$$S_m(y) = \overline{S}_\alpha(y)\overline{S}_\beta(y), \qquad \alpha+\beta=m, \tag{3.3}$$

$$S_m(y) = \frac{y+1}{2}\left[S_{m-1}^{(1)}(y) + S_{m-1}^{(2)}(y)\right] + \frac{y+3}{2}\left[S_{m-2}^{(1)}(y) + S_{m-2}^{(2)}(y)\right], \tag{3.4}$$

$$S_m(y) = \frac{y}{5}\left[\sum_{i=1}^{5} \overline{S}_{m-1}^{(i)}(y)\right] + \frac{2y+5}{5}\left[\sum_{i=1}^{5} \overline{S}_{m-2}^{(i)}(y)\right]. \tag{3.5}$$

By Euler's polyhedral formula we know that we have only the following (not mutually exclusive) four cases to consider:

Case I. P_{m+3} has a proper two-ring. In this case, we use (3.2) with our inductive hypothesis asserting that

$$(y+2)^{\gamma} \ll S_{\gamma}(y) \ll (y+3)^{\gamma}, \qquad \gamma = \alpha \text{ and } \beta.$$

Hence

$$(y+2)^{\alpha+\beta} \ll S_{\alpha}(y)S_{\beta}(y) \ll (y+3)^{\alpha+\beta}.$$

Multiplying by $(y+3)$ and using (3.2), we get

$$(y+3)(y+2)^{\alpha+\beta} \ll S_m(y) \ll (y+3)^{\alpha+\beta+1};$$

but, since $\alpha+\beta+1=m$ and $(y+2)^{\alpha+\beta+1} \ll (y+3)(y+2)^{\alpha+\beta}$, we obtain

$$(3.6) \qquad (y+2)^m \ll S_m(y) \ll (y+3)^m,$$

which is equivalent to (3.1).

Case II. P_{m+3} has a proper three-ring. In this case, we use (3.3) with our inductive hypothesis asserting that

$$(y+2)^{\gamma} \ll \overline{S}_{\gamma}(y) \ll (y+3)^{\gamma}, \qquad \gamma = \alpha \text{ and } \beta.$$

It follows that

$$(y+2)^m = (y+2)^{\alpha+\beta} \ll \overline{S}_{\alpha}(y)\overline{S}_{\beta}(y) = S_m(y) \ll (y+3)^m,$$

and (3.6) has been established for this case also.

Case III. P_{m+3} has a four-sided region surrounded by a proper four-ring. Hence we use (3.4). Our inductive hypothesis yields

$$(y+1)(y+2)^{m-1} \ll \frac{y+1}{2}\left[S_{m-1}^{(1)}(y) + S_{m-1}^{(2)}(y)\right] \ll (y+1)(y+3)^{m-1}$$

and

$$(y+3)(y+2)^{m-2} \ll \frac{y+3}{2}\left[S_{m-2}^{(1)}(y) + S_{m-2}^{(2)}(y)\right] \ll (y+3)^{m-1}.$$

Adding, we get in virtue of (3.4) the following:

$$(y^2+4y+5)(y+2)^{m-2} \ll S_m(y) \ll (y+2)(y+3)^{m-1}.$$

Since $(y+2)^2 \ll y^2+4y+5$ and $y+2 \ll y+3$, we see that (3.6) is again established.

Case IV. P_{m+3} has a five-sided region surrounded by a proper five-ring. Here we use (3.5). Our inductive hypothesis yields

$$y(y+2)^{m-1} \ll \frac{y}{5}\sum_{i=1}^{5} \overline{S}_{m-1}^{(i)}(y) \ll y(y+3)^{m-1}$$

and

$$(2y+5)(y+2)^{m-2} \ll \frac{2y+5}{5}\sum_{i=1}^{5} \overline{S}_{m-2}^{(i)}(y) \ll (2y+5)(y+3)^{m-2}.$$

Adding and using (3.5), we find that

$$(y^2 + 4y + 5)(y + 2)^{m-2} \ll S_m(y) \ll (y^2 + 5y + 5)(y + 3)^{m-2}.$$

Since $(y+2)^2 \ll y^2+4y+5$ and $(y^2+5y+5) \ll (y+3)^2$, we see that (3.6) has now been finally established.

In conclusion it may be noted that, while it is possible for $Q_n(\lambda)$ to equal either $(\lambda-2)^n$ or $(\lambda-3)^n$, the latter polynomial appears to approximate $Q_n(\lambda)$ the more closely for *regular* maps. This is one of the rough conclusions that can be drawn from the calculations of Chapter II and is undoubtedly connected with the experimentally verified fact that the successive derivatives of the chromatic polynomials for regular maps are numerically very small for $\lambda=3$ as compared with their values for either $\lambda=2$, where they alternate in sign in accordance with the results of Chapter III, or for $\lambda=4$, where, in accordance with our conjecture of §2, they are all positive. Unfortunately, we have not been able to express these results in precise form, much less to prove a rigorous theorem which would adequately cover the situation.

CHAPTER V. ANALYSES OF THE FOUR-RING AND FIVE-RING

1. **Formula for the reduction of the four-ring in terms of certain constrained chromatic polynomials.** Suppose we have a map P containing a 4-ring, R_1, R_2, R_3, R_4, with closed curve C. Let M^{in} [M^{ex}] represent the map obtained from P by replacing the inside [outside] of C by a single region Q^{in} [Q^{ex}], which is thus in contact with each of the four regions R_1, R_2, R_3, R_4, or at least with what remains of them. Let $A_1^{\text{in}}(\lambda)$ [$A_1^{\text{ex}}(\lambda)$] denote the number of ways that $M^{\text{in}}-Q^{\text{in}}$ [$M^{\text{ex}}-Q^{\text{ex}}$] can be colored so that four distinct colors occur in the ring. Let $A_2^{\text{in}}(\lambda)$ [$A_2^{\text{ex}}(\lambda)$] denote the numbers of ways that $M^{\text{in}}-Q^{\text{in}}$ [$M^{\text{ex}}-Q^{\text{ex}}$] can be colored so that R_1 and R_3 have the same color but R_2 and R_4 have distinct colors. Let $A_3^{\text{in}}(\lambda)$ [$A_3^{\text{ex}}(\lambda)$] denote the number so that R_1 and R_3 have distinct colors but R_2 and R_4 are colored alike; and finally let $A_4^{\text{in}}(\lambda)$ [$A_4^{\text{ex}}(\lambda)$] denote the number so that just two distinct colors appear in the ring. Then we obviously have

$$(1.1)\qquad \begin{aligned} P(\lambda) = {} & \frac{A_1^{\text{in}}(\lambda)A_1^{\text{ex}}(\lambda)}{\lambda(\lambda-1)(\lambda-2)(\lambda-3)} + \frac{A_2^{\text{in}}(\lambda)A_2^{\text{ex}}(\lambda) + A_3^{\text{in}}(\lambda)A_3^{\text{ex}}(\lambda)}{\lambda(\lambda-1)(\lambda-2)} \\ & + \frac{A_4^{\text{in}}(\lambda)A_4^{\text{ex}}(\lambda)}{\lambda(\lambda-1)}. \end{aligned}$$

But it is to be remembered that this formula (1.1) is still not analogous to formulas (7.1) and (7.2) of Chapter I for the 2- and 3-ring because the $A(\lambda)$'s are constrained polynomials rather than free polynomials. We therefore study the possibility of expressing the $A(\lambda)$'s in terms of certain free chromatic polynomials.

2. **Formula for the reduction of the four-ring in terms of free polynomials.** We can, of course, discuss the $A^{\text{in}}(\lambda)$'s and $A^{\text{ex}}(\lambda)$'s simultaneously. Accord-

ingly in the sequel we omit the superscripts. Let K_i be the map obtained from M by omitting the boundary between R_i and Q. Let L_i be the map obtained from M by omitting the boundaries between R_i and Q and between R_{i+2} and Q. Here subscripts are taken modulo 4. The four possibly topologically distinct maps thus obtained are illustrated in figures 1, 2, 3, 4, with figure 1 representing $K_1=K_3$, figure 2 representing $L_1=L_3$, figure 3 representing $K_2=K_4$, and with figure 4 representing $L_2=L_4$.

The four schemes represented by A_1, A_2, A_3, A_4 are illustrated in figure 24 below. A comparison of figure 24 with figures 1–4 leads at once to the follow-

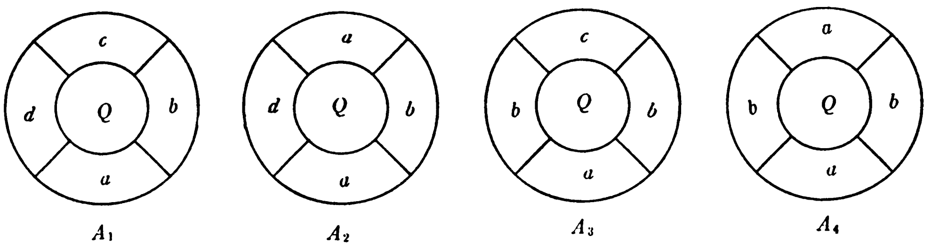


FIG. 24

ing four equations involving the chromatic polynomials $K_i(\lambda)$ and $L_i(\lambda)$:

$$(2.1)\qquad \begin{aligned} K_1(\lambda) &= A_1(\lambda) + A_3(\lambda), & K_2(\lambda) &= A_1(\lambda) + A_2(\lambda),\\ L_1(\lambda) &= A_2(\lambda) + A_4(\lambda), & L_2(\lambda) &= A_3(\lambda) + A_4(\lambda), \end{aligned}$$

which we would like to solve for the A's in terms of the K's and L's. These equations are, however, not independent, a fact which is not surprising in view of the equality $K_1(\lambda)+L_1(\lambda)=K_2(\lambda)+L_2(\lambda)$, which is merely (1.3) of Chapter I in another notation. Hence, before we can solve (2.1), we must find an additional equation. The deficiency is eliminated by the following:

$$(2.2)\qquad A_1(\lambda) + (\lambda - 2)(\lambda - 3)A_4(\lambda) = (\lambda - 3)[A_2(\lambda) + A_3(\lambda)].$$

We have two quite different proofs of this fundamental relation which we defer to §§3 and 4. We merely note here that the simultaneous solution of (2.1) and (2.2) leads to the following four equalities:

$$(2.3)\qquad \begin{aligned} (\lambda^2-3\lambda+1)A_1(\lambda) &= \lambda(\lambda-3)K_1(\lambda) + (\lambda-3)L_1(\lambda) - (\lambda-3)(\lambda-1)L_2(\lambda),\\ (\lambda^2-3\lambda+1)A_2(\lambda) &= K_1(\lambda) + (\lambda-2)^2L_1(\lambda) - (\lambda-2)L_2(\lambda),\\ (\lambda^2-3\lambda+1)A_3(\lambda) &= K_1(\lambda) - (\lambda-3)L_1(\lambda) + (\lambda-3)(\lambda-1)L_2(\lambda),\\ (\lambda^2-3\lambda+1)A_4(\lambda) &= -K_1(\lambda) + (\lambda-3)L_1(\lambda) + (\lambda-2)L_2(\lambda). \end{aligned}$$

The substitution of the value for the $A(\lambda)$'s given by (2.3) in equation (1.1) gives us a formula for the reduction of the 4-ring in terms of free chromatic polynomials associated with maps having fewer regions than P has.

3. **Proof of** (2.2) **using Kempe chains.** Assume temporarily that $\lambda \geqq 4$. From our λ colors select a particular set of four distinct colors a, b, c, d. Let $A_1^*, A_2^*, A_3^*, A_4^*$ denote the number of ways of coloring $M-Q$ in such a way that these four colors appear in the ring in exactly the manner indicated in the four schemes of figure 24. Since it is possible to select the four colors a, b, c, d, in the indicated order, in just $\lambda(\lambda-1)(\lambda-2)(\lambda-3)$ ways, it is clear that

$$A_1(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)A_1^*. \tag{3.1}$$

Similarly it is possible to select a, b, c in just $\lambda(\lambda-1)(\lambda-2)$ ways, and hence

$$A_2(\lambda) = \lambda(\lambda-1)(\lambda-2)A_2^*, \tag{3.2}$$

$$A_3(\lambda) = \lambda(\lambda-1)(\lambda-2)A_3^*. \tag{3.3}$$

Similarly, we have

$$A_4(\lambda) = \lambda(\lambda-1)A_4^*. \tag{3.4}$$

Now we think of our λ colors as divided into two *complementary* sets, the first set containing colors a, c, and perhaps some other colors, the second set containing colors b, d, and perhaps some others. Let $A_i^*(ac)$ denote the number of colorations enumerated in A_i^* such that R_1 and R_3 are connected by a set of regions colored solely in the set $a, c, \cdots$. Let $A_i^*(bd)$ denote the number of colorations enumerated in A_i^* such that R_2 and R_4 are connected by a set of regions colored solely in the set $b, d, \cdots$. Then by the well known principles of Kempe chains the following relations are obvious:

$$A_i^* = A_i^*(ac) + A_i^*(bd), \qquad i = 1, 2, 3, 4, \tag{3.5}$$

$$A_1^*(ac) = A_3^*(ac), \qquad A_1^*(bd) = A_2^*(bd),$$

$$A_4^*(ac) = A_2^*(ac), \qquad A_4^*(bd) = A_3^*(bd).$$

Adding these last four equations and making use of (3.5) we obtain

$$A_1^* + A_4^* = A_2^* + A_3^*. \tag{3.6}$$

Finally, eliminating $A_1^*, A_2^*, A_3^*, A_4^*$ from equations (3.1), (3.2), (3.3), (3.4), and (3.6), we get

$$\frac{A_1(\lambda)}{\lambda(\lambda-1)(\lambda-2)(\lambda-3)} + \frac{A_4(\lambda)}{\lambda(\lambda-1)} = \frac{A_2(\lambda) + A_3(\lambda)}{\lambda(\lambda-1)(\lambda-2)}.$$

Clearing of fractions we obtain the desired equation (2.2) which has thus been proved for all integral values of $\lambda \geqq 4$. Since, however, both sides of (2.2) are polynomials, it is clear that (2.2) is an identity in λ.

4. **Proof of** (2.2) **by induction.** In the previous sections we have been discussing a given map M of triple vertices having a quadrilateral Q surrounded by a (not necessarily proper) 4-ring, R_1, R_2, R_3, R_4. There are just four essentially([20]) distinct proper maps of this type having no region completely exterior to the ring. We denote them by M^1, M^2, M^3, M^4 and describe them as follows:

M^1 is the map of five regions in which R_1 has contact with R_3.

M^2 is the map of five regions in which R_2 has contact with R_4.

M^3 is the map of four regions in which R_1 is identical with R_3.

M^4 is the map of four regions in which R_2 is identical with R_4.

These maps are illustrated in figure 25.

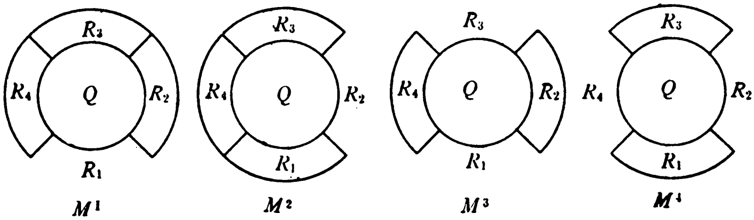


FIG. 25

Let the constrained chromatic polynomials $A_i^j(\lambda)$ be defined with reference to M^j exactly as $A_i(\lambda)$ was defined with reference to M. On account of the simple nature of M^j, it is easy to write down explicitly the values of the $A_i^j(\lambda)$. From their definitions, we find at once that

$$(4.1)\quad \begin{array}{llll} A_1^1=\lambda(\lambda-1)(\lambda-2)(\lambda-3), & A_2^1=0, & A_3^1=\lambda(\lambda-1)(\lambda-2), & A_4^1=0, \\ A_1^2=\lambda(\lambda-1)(\lambda-2)(\lambda-3), & A_2^2=\lambda(\lambda-1)(\lambda-2), & A_3^2=0, & A_4^2=0, \\ A_1^3=0, & A_2^3=\lambda(\lambda-1)(\lambda-2), & A_3^3=0, & A_4^3=\lambda(\lambda-1), \\ A_1^4=0, & A_2^4=0, & A_3^4=\lambda(\lambda-1)(\lambda-2), & A_4^4=\lambda(\lambda-1). \end{array}$$

From the italicized remark at the end of the proof of Theorem I, §6, Chapter I([21]), it is apparent that there exist relations of the type:

$$(4.2)\qquad A_i(\lambda) = \sum_{j=1}^{4} B_j(\lambda)A_i^j(\lambda), \qquad i = 1, 2, 3, 4,$$

([20]) In case the reader has trouble with the meaning of the word "essential" the following will suffice for the present: Two maps of the type under discussion are essentially the same if (1) they are topologically equivalent and if (2) the homeomorphism which establishes this equivalence can be chosen so that the regions marked Q and R_i of one map correspond respectively to the regions marked Q and R_i of the other map for each i ($i=1, 2, 3, 4$). For a more formal treatment of this matter see §1 of Chapter VI.

([21]) The notation is different from that of §6, Chapter I. Also what was there referred to as the interior of the ring is now thought of as the exterior, and vice versa.

where $B_j(\lambda)$ is a polynomial in λ, independent of i but dependent on the particular map M under discussion. Hence, if there also exist relations of the type

$$\sum_{i=1}^{4} a_i(\lambda)A_i^j(\lambda) = 0 \qquad \text{for } j = 1, 2, 3, 4, \tag{4.3}$$

where a_i is independent of j (and also of M), it follows from (4.2) that we must also have

$$\sum_{i=1}^{4} a_i(\lambda)A_i(\lambda) = 0 \tag{4.4}$$

for any map M. We proceed to determine a's that will indeed satisfy (4.3) and thus obtain a relation of the type (4.4) valid for any map M.

Using (4.1) and suppressing a common factor $\lambda(\lambda-1)$, we may write equations (4.3) as follows:

$$\begin{aligned}
(\lambda-2)(\lambda-3)a_1 + \quad 0\cdot a_2 + (\lambda-2)a_3 + 0\cdot a_4 &= 0,\\
(\lambda-2)(\lambda-3)a_1 + (\lambda-2)a_2 + \quad 0\cdot a_3 + 0\cdot a_4 &= 0,\\
0\cdot a_1 + (\lambda-2)a_2 + \quad 0\cdot a_3 + 1\cdot a_4 &= 0,\\
0\cdot a_1 + \quad 0\cdot a_2 + (\lambda-2)a_3 + 1\cdot a_4 &= 0.
\end{aligned}$$

These homogeneous equations in the four unknowns a_1, a_2, a_3, a_4 are compatible and have the essentially unique solution,

$$a_1 = -1, \qquad a_2 = \lambda - 3, \qquad a_3 = \lambda - 3, \qquad a_4 = -(\lambda-2)(\lambda-3).$$

Substituting these values for the a's in (4.4), we see that we have again derived the relation (2.2) and that it has now been established by an inductive process. The induction appears explicitly in the proof of Theorem I, §6, Chapter I.

The method of §3 for deriving (2.2) is somewhat shorter than the method of §4, but it was the latter method which we first discovered.

5. **Four-color reducibility of the four-ring.** We wish to point out in this section that the relation (2.2) contains essentially Birkhoff's earlier result on the reducibility of the proper four-ring (cf. Birkhoff [1]). The essential facts needed are formulated in the following theorem:

THEOREM I. *If* $L_1(4) > 0$ *and* $L_2(4) > 0$, *then at least two of the three quantities* $A_2(4)$, $A_3(4)$, *and* $A_4(4)$ *must be positive.*

Proof. If $A_4(4) > 0$, (2.2) shows that either $A_2(4)$ or $A_3(4)$ or both must be positive and the theorem is true. The only other possibility is for $A_4(4)$ to vanish. But, in this case, (2.1) shows that both $A_2(4)$ and $A_3(4)$ must be positive, and the theorem is therefore completely established.

The 4c. reducibility of the proper 4-ring is an immediate corollary of this theorem. For, if the map P of §1 were 4c. irreducible, $L_1^{in}(4)>0$, $L_2^{in}(4)>0$, $L_1^{ex}(4)>0$, and $L_2^{ex}(4)>0$, inasmuch as each of the four maps just referred to has fewer regions than P. Hence by the theorem just proved two of the quantities $A_2^{in}(4)$, $A_3^{in}(4)$, $A_4^{in}(4)$ are positive and also two of the quantities $A_2^{ex}(4)$, $A_3^{ex}(4)$, $A_4^{ex}(4)$. It follows from (1.1) that $P(4)>0$, contradicting the assumption that P was 4c. irreducible.

It is easy to find simple examples which show that any one of the quantities $A_1(4)$, $A_2(4)$, $A_3(4)$, $A_4(4)$ may vanish, even if we assume that R_1, R_2, R_3, R_4 form a *proper* ring. Nevertheless, in addition to the information already given in Theorem I, it is easy to prove the following theorem.

THEOREM II. *If $K_1(4)$ and $K_2(4)$ are positive as well as $L_1(4)$ and $L_2(4)$, then at least three of the quantities $A_1(4)$, $A_2(4)$, $A_3(4)$, $A_4(4)$ are positive.*

Proof. If $A_2(4)$, $A_3(4)$, $A_4(4)$ are all positive, there is nothing to be proved. But by Theorem I, we know that not more than one of these quantities can vanish. There are thus only the following three possibilities to be considered:

I. If $A_2(4)=0$, then the second of equations (2.1) shows that $A_1(4)>0$.

II. If $A_3(4)=0$, then the first of equations (2.1) shows that $A_1(4)>0$.

III. If $A_4(4)=0$, then equation (2.2) shows that $A_1(4)=A_2(4)+A_3(4)>0$.

In all three of these cases the theorem is seen to be true.

6. **Inequalities satisfied by $K_1(\lambda)$, $K_2(\lambda)$, $L_1(\lambda)$, $L_2(\lambda)$.** From the fact that $A_i(\lambda)\geqq 0$ for $\lambda=0, 1, 2, \cdots$ and that $\lambda^2-3\lambda+1\geqq 0$ for $\lambda\geqq 2^{-1}(3+5^{1/2})$, it is clear, from (2.3) and the relation $K_1(\lambda)+L_1(\lambda)=K_2(\lambda)+L_2(\lambda)$, that we can immediately write down numerous inequalities involving the four quantities $K_1(\lambda)$, $K_2(\lambda)$, $L_1(\lambda)$, $L_2(\lambda)$ valid for $\lambda=3, 4, \cdots$. In this connection we make two remarks, the first of which lends significance to these inequalities and the second of which is designed to show, among other things, that these inequalities are valid also for non-integral values of $\lambda>5$.

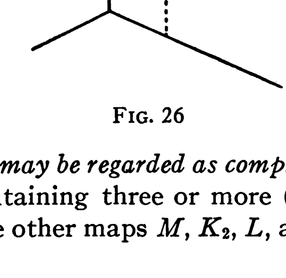


FIG. 26

Remark 1. *The map K_1 may be regarded as completely arbitrary.* In fact, any map of triple vertices containing three or more (simply connected) regions can be denoted by K_1. The other maps M, K_2, L, and L_2 are then obtained as follows: To get M, we draw a new boundary line l^1 (see dotted line in figure 26), sufficiently near any given boundary line l, so that l and l^1 are the opposite sides of a new quadrilateral Q. The maps K_2, L_1, and L_2 are then obtained from M, as in §2, by omitting various boundaries of Q.

Remark 2. Not only is $\lambda^2-3\lambda+1$ completely monotonic for $\lambda \geqq 3$ but *so also are* $A_1(\lambda)$, $A_2(\lambda)$, $A_3(\lambda)$, $A_4(\lambda)$ *for* $\lambda \geqq 5$.

The proof of the italicized statement is carried out by induction: Firstly, it is clear from (4.1) that the statement is true for the maps of fewest regions, that is, those maps which have only the region Q and certain other regions abutting Q. Secondly, it is clear from Euler's polyhedral formula that if the map possesses regions (other than Q) not abutting Q, at least one of these regions is a pentagon, quadrilateral, triangle, or two-sided region, and hence formulas (3.1), (2.1), (1.2), and (1.1) of Chapter I as applied to the present constrained polynomials are available for proving that if the statement is true for maps having less than n regions it must also be true for maps having n regions. The details, which are very similar to those of §3, Chapter IV (cf. also Birkhoff [4, p. 13]), will be omitted. Notice that it is not even necessary to guard against the occurrence of pseudo-maps in using the reduction formula for the pentagon or quadrilateral. For a pseudo-map is regarded as having a chromatic polynomial which vanishes identically and is hence completely monotonic for $\lambda \geqq 5$.

Of the various inequalities deducible from (2.3), perhaps the two most interesting ones are the following:

$$-\lambda(\lambda-3)K_1(\lambda) \leqq (\lambda-3)L_1(\lambda)-(\lambda-3)(\lambda-1)L_2(\lambda) \leqq K_1(\lambda). \tag{6.1}$$

According to the preceding discussion, (6.1) holds for $\lambda=3, 4$, and for $\lambda \geqq 5$. Moreover

$$\begin{aligned} -\frac{d^n}{d\lambda^n}\left[\lambda(\lambda-3)K_1(\lambda)\right] &\leqq \frac{d^n}{d\lambda^n}\left[(\lambda-3)L_1(\lambda)-(\lambda-3)(\lambda-1)L_2(\lambda)\right] \\ &\leqq \frac{d^n}{d\lambda^n}K_1(\lambda) \end{aligned} \tag{6.2}$$

for $\lambda \geqq 5$ and for $n=0, 1, 2, \cdots$.

If $K_1(4)=0$, we get from (6.1) the result that $L_1(4)=3L_2(4)$; and since $K_2=K_1+L_1-L_2=0+3L_2-L_2=2L_2(4)$, we come out with the interesting result that, if K_1 is 4c. irreducible, K_2 is not 4c. irreducible. This special result can, however, be obtained more simply without reference to (6.1) by a direct study of Kempe chains.

7. **Formula for the reduction of the five-ring in terms of constrained chromatic polynomials.** Suppose we have a map P containing a 5-ring, R_1, R_2, R_3, R_4, R_5, with closed curve C. Let M^{in} $[M^{\text{ex}}]$ represent the map obtained from P by replacing the inside [outside] of C by a single region Q^{in} $[Q^{\text{ex}}]$,

which is thus in contact with each of the five regions, $R_1, \cdots, R_5$, or at least with what remains of them. Let $G^{in}(\lambda)$ $[G^{ex}(\lambda)]$ denote the number of ways that $M^{in}-Q^{in}$ $[M^{ex}-Q^{ex}]$ can be colored so that five distinct colors occur in the ring. Let $A_i^{in}(\lambda)$ $[A_i^{ex}(\lambda)]$ denote the number of ways that $M^{in}-Q^{in}$ $[M^{ex}-Q^{ex}]$ can be colored so that R_{i-1} and R_{i+1} have the same color but the other three R's have three other distinct colors. The subscripts are here, and in the sequel, taken modulo 5 so that $i\equiv 1, 2, 3, 4, 5 \pmod 5$. Finally let $B_i^{in}(\lambda)$ $[B_i^{ex}(\lambda)]$ denote the number of ways that $M^{in}-Q^{in}$ $[M^{ex}-Q^{ex}]$ can be colored so that just three distinct colors appear in the ring and the color of R_i is distinct from the colors of the other four R's. It is readily seen that we have now exhausted the possible color schemes. Consequently we arrive at the following formula, which is for the five-ring exactly what (1.1) was for the four-ring:

$$P(\lambda) = \frac{G^{in}(\lambda)G^{ex}(\lambda)}{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)} \tag{7.1}$$
$$+ \frac{\sum_{i=1}^{5} A_i^{in}(\lambda)A_i^{ex}(\lambda)}{\lambda(\lambda-1)(\lambda-2)(\lambda-3)} + \frac{\sum_{i=1}^{5} B_i^{in}(\lambda)B_i^{ex}(\lambda)}{\lambda(\lambda-1)(\lambda-2)}.$$

We next study the possibility of expressing the constrained chromatic polynomials, $G(\lambda)$, $A_i(\lambda)$, $B_i(\lambda)$, in terms of certain free chromatic polynomials.

8. **Formula for the reduction of the five-ring in terms of free polynomials.** Since we can discuss M^{in} and M^{ex} simultaneously, we shall hereafter omit the superscripts.

Let K_i be the map obtained from M by erasing the boundary between R_i and Q. Let L_i be the map obtained from M by erasing the boundary between R_{i-1} and Q and the boundary between R_{i+1} and Q. By Fundamental Principle (1.3) of Chapter I, it may be shown that

$$K_i(\lambda) + L_{i+1}(\lambda) = L_{i+2}(\lambda) + K_{i+3}(\lambda). \tag{8.1}$$

Referring to the definitions of the A's, B's, and G, we see that

$$K_i(\lambda) = A_i(\lambda) + A_{i+2}(\lambda) + A_{i+3}(\lambda) + B_i(\lambda) + G(\lambda). \tag{8.2}$$

$$L_i(\lambda) = A_i(\lambda) + B_{i+2}(\lambda) + B_{i+3}(\lambda); \qquad i = 1, 2, 3, 4, 5 \pmod 5. \tag{8.3}$$

It is clear from (8.1) that the ten equations (8.2) and (8.3) are not independent. Hence, before we can solve the equations (8.2) and (8.3) for the unknown A's, B's and G, we must find some additional equations. The deficiency is just eliminated by the following five equations:

$$G(\lambda) + (\lambda-3)(\lambda-4)B_{i+2}(\lambda) = (\lambda-4)(A_i(\lambda) + A_{i+4}(\lambda)), \qquad i \equiv 1, 2, 3, 4, 5. \tag{8.4}$$

These equations are analogous to the single equation (2.2) which appeared in the analysis of the four-ring. The two proofs of (8.4) are postponed to §§9 and 10. We outline here the solution of equations (8.2), (8.3), and (8.4) for the A's, B's and G in terms of K_1, L_1, L_2, L_3, L_4, L_5. First, with the help of (8.4) we eliminate G, A_2, A_3, A_4, A_5 from (8.2) and (8.3) and thus obtain the following system of six independent equations, in which we have put $u=\lambda-3$:

$$(8.5)\quad \begin{aligned}
&(2u+1)A_1(\lambda)+(-u^2+u+1)B_1(\lambda)+u^2B_2(\lambda)-u^2B_3(\lambda)-u^2B_4(\lambda)+u^2B_5(\lambda)=K_1(\lambda),\\
&A_1(\lambda)+B_3(\lambda)+B_4(\lambda)=L_1(\lambda),\\
&A_1(\lambda)-uB_1(\lambda)+uB_2(\lambda)-uB_3(\lambda)+B_4(\lambda)+(u+1)B_5(\lambda)=L_2(\lambda),\\
&A_1(\lambda)+B_1(\lambda)-uB_4(\lambda)+(u+1)B_5(\lambda)=L_3(\lambda),\\
&A_1(\lambda)+B_1(\lambda)+(u+1)B_2(\lambda)-uB_3(\lambda)=L_4(\lambda),\\
&A_1(\lambda)-uB_1(\lambda)+(u+1)B_2(\lambda)+B_3(\lambda)-uB_4(\lambda)+uB_5(\lambda)=L_5(\lambda).
\end{aligned}$$

We next solve these equations for $A_1(\lambda)$, $B_1(\lambda)$, $B_2(\lambda)$, $B_3(\lambda)$, $B_4(\lambda)$, and $B_5(\lambda)$. We thus obtain the first six of the following eleven equations:

$$(8.6)\quad \begin{aligned}
(u^2+3u+1)B_1(\lambda)&=-K_1(\lambda)+L_1(\lambda)-L_2(\lambda)+(u+1)L_3(\lambda)\\
&\quad+(u+1)L_4(\lambda)-L_5(\lambda),\\
(u^2+3u+1)B_2(\lambda)&=-K_1(\lambda)+(u+1)L_4(\lambda)+uL_5(\lambda),\\
(u^2+3u+1)B_3(\lambda)&=-K_1(\lambda)+(u+1)L_1(\lambda)-L_2(\lambda)+L_3(\lambda)+uL_5(\lambda),\\
(u^2+3u+1)B_4(\lambda)&=-K_1(\lambda)+(u+1)L_1(\lambda)+uL_2(\lambda)+L_4(\lambda)-L_5(\lambda),\\
(u^2+3u+1)B_5(\lambda)&=-K_1(\lambda)+uL_2(\lambda)+(u+1)L_3(\lambda),\\
(u^2+3u+1)A_1(\lambda)&=2K_1(\lambda)+(u^2+u-1)L_1(\lambda)-(u-1)L_2(\lambda)-L_3(\lambda)\\
&\quad-L_4(\lambda)-(u-1)L_5(\lambda),\\
(u^2+3u+1)A_2(\lambda)&=2K_1(\lambda)-(u+1)L_1(\lambda)+(u^2+u+1)L_2(\lambda)\\
&\quad-(u+1)L_3(\lambda)-L_4(\lambda)+L_5(\lambda),\\
(u^2+3u+1)A_3(\lambda)&=2K_1(\lambda)-L_1(\lambda)-(u-1)L_2(\lambda)+(u^2+u-1)L_3(\lambda)\\
&\quad-(u+1)L_4(\lambda)+L_5(\lambda),\\
(u^2+3u+1)A_4(\lambda)&=2K_1(\lambda)-L_1(\lambda)+L_2(\lambda)-(u+1)L_3(\lambda)\\
&\quad+(u^2+u-1)L_4(\lambda)-(u-1)L_5(\lambda),\\
(u^2+3u+1)A_5(\lambda)&=2K_1(\lambda)-(u+1)L_1(\lambda)+L_2(\lambda)-L_3(\lambda)\\
&\quad-(u+1)L_4(\lambda)+(u^2+u+1)L_5(\lambda),\\
(u^2+3u+1)G(\lambda)&=(u-1)\left[(u+4)K_1(\lambda)-(u+2)\{L_1(\lambda)+L_3(\lambda)+L_4(\lambda)\}\right.\\
&\quad\left.+2\{L_2(\lambda)+L_5(\lambda)\}\right]
\end{aligned}$$
[22].

---

[22] Since the regions R_1, R_2, R_3, R_4, R_5 can be renamed R_1, R_5, R_4, R_3, R_2, it is clear that equations (8.5) and (8.6) should be invariant as a set under the permutation (25)(34). This serves as a check on our calculations.

The seventh, eighth, ninth and tenth of equations (8.6) may be found by advancing the subscripts of the sixth equation and eliminating K_2, K_3, K_4, K_5 with the help of (8.1). The last equation can be deduced from the equations previously found in conjunction with any one of the equations (8.4).

The substitution of the values of $G(\lambda)$, $A_i(\lambda)$, and $B_i(\lambda)$ given by (8.6) into equation (7.1) yields a formula for the reduction of the 5-ring in terms of free chromatic polynomials associated with maps having fewer regions than P has.

9. **Proof of (8.4) using Kempe chains.** Assume temporarily that $\lambda \geqq 5$. From our λ colors select a particular set of five distinct colors, a, b, c, d, e. Let G^*, A_i^*, A_{i+4}^*, B_{i+2}^* denote the number of ways of coloring $M-Q$ in such a way that the five regions surrounding Q are colored in the manner indicated in the following table:

	R_i	R_{i+1}	R_{i+2}	R_{i+3}	R_{i+4}
G^*	a	b	c	d	e
A_i^*	a	b	c	d	b
A_{i+4}^*	a	b	c	a	e
B_{i+2}^*	a	b	c	a	b

(i is regarded as fixed). Since the three colors a, b, c can be chosen from the λ colors in the indicated order in $\lambda(\lambda-1)(\lambda-2)$ ways, it is clear that

$$(9.1) \qquad B_{i+2}(\lambda) = \lambda(\lambda-1)(\lambda-2)B_{i+2}^*.$$

Similarly, we have

$$(9.2) \qquad A_k(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)A_k^*, \qquad k = i \text{ or } i+4,$$

and

$$(9.3) \qquad G(\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)G^*.$$

Now we think of our λ colors as divided into two complementary sets; the first set containing colors a, c, d and perhaps some others; the second set containing colors b, e, and perhaps some others. Let $G^*(acd)$, $A_k^*(acd)$, and $B_{i+2}^*(acd)$ denote the number of colorations enumerated by G^*, A_k^*, and B_{i+2}^*, respectively, such that the regions R_i, R_{i+2}, R_{i+3} are connected by a set of regions colored solely in the set, $a, c, d, \cdots$. Let $G^*(be)$, $A_k^*(be)$, $B_{i+2}^*(be)$ denote the number of colorations enumerated by G^*, A_k^*, and B_{i+2}^*, respectively, such that the regions R_{i+1} and R_{i+4} are connected by a set of regions colored solely in the set, $b, e, \cdots$. The following relations are then obvious:

$$G^* = G^*(acd) + G^*(be),$$
(9.4)
$$A_k^* = A_k^*(bcd) + A_k^*(be),$$
$$B_{i+2}^* = B_{i+2}^*(acd) + B_{i+2}^*(be),$$

$$G^*(acd) = A_i^*(acd),$$
$$G^*(be) = A_{i+4}^*(be),$$
$$B_{i+2}^*(be) = A_i^*(be),$$
$$B_{i+2}^*(acd) = A_{i+4}^*(acd).$$

Adding these last four equations and making use of (9.4) we obtain

(9.5) $$G^* + B_{i+2}^* = A_i^* + A_{i+4}^*.$$

Finally, eliminating the starred quantities with the help of (9.1), (9.2), and (9.3), we have

$$\frac{G(\lambda)}{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)} + \frac{B_{i+2}(\lambda)}{\lambda(\lambda-1)(\lambda-2)} = \frac{A_i(\lambda) + A_{i+4}(\lambda)}{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}.$$

Clearing of fractions we obtain the desired equation (8.4), which has thus been established for integral values of $\lambda \geqq 5$. Since, however, both sides of (8.4) are polynomials, it is clear that (8.4) is an identity in λ.

It is interesting to note that the five linearly independent relations (8.4) are of no significance for the interesting case $\lambda = 4$, for which they reduce to $0 = 0$. But for $\lambda \neq 4$, we can eliminate G, thus obtaining the relations

(9.6) $$A_i(\lambda) + (\lambda - 3)B_{i+1}(\lambda) = (\lambda - 3)B_{i+2}(\lambda) + A_{i+3}(\lambda), \quad i = 1, 2, 3, 4, 5$$ [23].

Since, however, both sides of each of these equalities are polynomials, it is clear that (9.6) holds also for $\lambda = 4$. This gives us a simple example of how function-theoretic considerations can give significant results for the case of special interest, $\lambda = 4$. The function-theoretic method is, however, not essential in this connection, as it is also possible to establish (9.6) directly, either by using Kempe chains, or by using the inductive method of the next section.

10. **Proof of (8.4) by induction.** In §§8, 9 we have been considering a given map M of triple vertices having a pentagon Q surrounded by a (not necessarily proper) 5 ring, R_1, R_2, R_3, R_4, R_5. There are just ten essentially distinct maps of this type having no region completely exterior to the ring. They are: M_i, in which R_{i-1} and R_{i+1} are identical; and N_i, in which the five regions of the circuit are distinct but R_i has contact with each of the other four. Let

[23] Any four of these five relations are linearly independent.

the constrained polynomials $GM_j(\lambda)$, $A_iM_j(\lambda)$, $B_iM_j(\lambda)$ be defined with reference to M_j exactly as $G(\lambda)$, $A_i(\lambda)$, $B_i(\lambda)$ were defined with reference to M. Similarly we let $GN_j(\lambda)$, $A_iN_j(\lambda)$, $B_iN_j(\lambda)$ equal the number of ways N_j-Q can be colored with λ colors in the schemes indicated by G, A_i, B_i respectively.

We first wish to solve the homogeneous linear equations

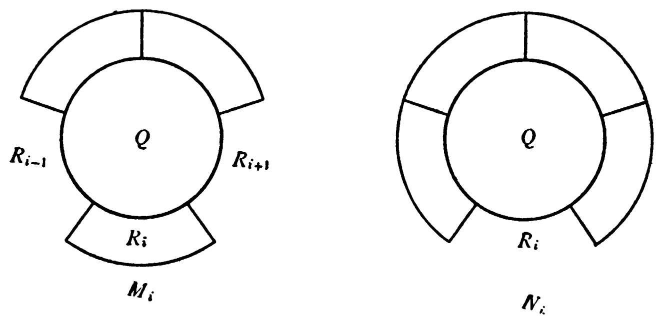


FIG. 27

$$(10.1)\quad \begin{aligned} &\sum_{i=1}^{5} a_iA_iM_j(\lambda) + \sum_{i=1}^{5} b_iB_iM_j(\lambda) + gGM_j(\lambda) = 0,\\ &\sum_{i=1}^{5} a_iA_iN_j(\lambda) + \sum_{i=1}^{5} b_iB_iN_j(\lambda) + gGN_j(\lambda) = 0, \qquad j = 1, 2, 3, 4, 5, \end{aligned}$$

for the unknowns $a_1, \cdots, a_5, b_1, \cdots, b_5, g$. For we know from the italicized remark at the end of the proof of Theorem I, §6, Chapter I, that there exist relations of the type,

$$A_i(\lambda) = \sum_{j=1}^{5} c_j(\lambda)A_iM_j(\lambda) + \sum_{j=1}^{5} d_j(\lambda)A_iN_j(\lambda),$$

$$B_i(\lambda) = \sum_{j=1}^{5} c_j(\lambda)B_iM_j(\lambda) + \sum_{j=1}^{5} d_j(\lambda)B_iN_j(\lambda),$$

$$G(\lambda) = \sum_{j=1}^{5} c_j(\lambda)GM_j(\lambda) + \sum_{j=1}^{5} d_j(\lambda)GN_j(\lambda),$$

and these identities in conjunction with (10.1) would yield

$$(10.2)\quad \sum_{i=1}^{5} a_iA_i(\lambda) + \sum_{i=1}^{5} b_iB_i(\lambda) + gG(\lambda) = 0,$$

which is the type of result we are seeking. Now the equations (10.1) are fortunately not independent. Indeed our Fundamental Principle (1.3) of Chapter I yields

$$GN_j(\lambda) + GM_{j+1}(\lambda) = GM_{j+2}(\lambda) + GN_{j+3}(\lambda),$$
$$A_iN_j(\lambda) + A_iM_{j+1}(\lambda) = A_iM_{j+2}(\lambda) + A_iN_{j+3}(\lambda),$$
$$B_iN_j(\lambda) + B_iM_{j+1}(\lambda) = B_iM_{j+2}(\lambda) + B_iN_{j+3}(\lambda), \qquad j = 1, \cdots, 5.$$

Hence we have a total of at least five linearly independent solutions. If the rank of the matrix of the coefficients in (10.1) is exactly 6, as it turns out to be, we can solve for a properly chosen six of the unknowns in terms of the other five. A particular solution may be found by assigning special numerical values to these five. It is easy to calculate the coefficients in (10.1). In fact the first of these equations may be written, after suppressing a factor $\lambda(\lambda-1)(\lambda-2)$, as follows:

$$(\lambda - 3)a_1 + b_3 + b_4 = 0. \tag{10.3}$$

Similarly, the first of the second group of equations (10.1) may be written

$$(\lambda - 3)(a_1 + a_3 + a_4) + b_1 + (\lambda - 3)(\lambda - 4)g = 0 \tag{10.4}$$

and the other eight equations may be obtained by advancing the subscripts (mod 5) in (10.3) and (10.4). We seek a solution of these equations in which (say) $a_2=a_3=a_4=a_5=0$ and $g=1$. Advancing the subscripts in (10.3) we find immediately that $b_1=b_3=b_4=-b_2=-b_5$. Adding 1 to the subscripts of (10.4), we get $b_2=-(\lambda-3)(\lambda-4)$, and (10.4) itself will now yield $a_1=-2(\lambda-4)$.

To recapitulate, we have found the following solution of equations (10.1): $a_1=-2(\lambda-4)$, $a_2=a_3=a_4=a_5=0$, $b_1=b_3=b_4=(\lambda-3)(\lambda-4)$, $b_2=b_5=-(\lambda-3)(\lambda-4)$, $g=1$; and four other linearly independent solutions may be found by advancing subscripts. Substituting the above values for the a's, b's, and g in (10.2) we get

$$\begin{aligned} -2(\lambda - 4)A_1(\lambda) + (\lambda - 3)(\lambda - 4)[B_1(\lambda) - B_2(\lambda) + B_3(\lambda) \\ + B_4(\lambda) - B_5(\lambda)] + G(\lambda) = 0. \end{aligned} \tag{10.5}$$

Adding 4 (mod 5) to the subcripts, we get

$$\begin{aligned} -2(\lambda - 4)A_5(\lambda) + (\lambda - 3)(\lambda - 4)[B_5(\lambda) - B_1(\lambda) + B_2(\lambda) \\ + B_3(\lambda) - B_4(\lambda)] + G(\lambda) = 0. \end{aligned}$$

Adding to (10.5) and dividing by 2, we finally obtain

$$-(\lambda - 4)[A_1(\lambda) + A_5(\lambda)] + (\lambda - 3)(\lambda - 4)B_3(\lambda) + G(\lambda) = 0,$$

from which (8.4) follows at once.

The fact that we have *deduced* (8.4) may seem to be inconsistent with the heading of this section in which we refer to the proof of (8.4) by *induction*. The heading was so chosen because of the fact that the central principle used in the preceding argument is contained in Theorem I, §6, Chapter I, and this was established previously by induction.

The advantage of this method of deduction is that we see from the matrix of coefficients of (10.1) that the five equations (8.4) comprise essentially *all* relations of the type (10.2) which hold for any map M of the type under discussion.

11. **Four-color reducibility of the five-ring surrounding more than a single region.** We show in this section that the relations (9.6), and hence the relations (8.4) from which they are derived, contain the known result that a proper 5-ring surrounding more than a single region on each side is 4c. reducible. The essential facts are formulated in the following Theorem I, the proof of which is merely an adaptation of Birkhoff's earlier work on the five-ring (Birkhoff [1, pp. 120–122]). We first need the following lemmas, the first two of which are obvious consequences of (9.6) and the last of which follows equally obviously from (8.3).

LEMMA 1. *If* $B_i(4)>0$, *then either* $B_{i+1}(4)>0$ *or* $A_{i+2}(4)>0$, *or both are greater than* 0. *Likewise, if* $B_i>0$, *either* $B_{i-1}>0$ *or* $A_{i-2}>0$, *or both are greater than* 0.

LEMMA 2. *If* $A_i>0$, *then either* $B_{i+2}>0$ *or* $A_{i-2}>0$, *or both are greater than* 0. *Likewise, if* $A_i>0$, *either* $B_{i-2}>0$ *or* $A_{i+2}>0$, *or both are greater than* 0.

LEMMA 3. *If* $L_i>0$, *at least one of the quantities* A_i, B_{i+2}, B_{i-2} *is greater than* 0.

THEOREM I. *If all the L's are positive, one of the following three possibilities must hold*:

(I) $A_1=A_2=A_3=A_4=A_5=0$, $B_1=B_2=B_3=B_4=B_5>0$.

(II) $A_1=A_2=A_3=A_4=A_5>0$, $B_1=B_2=B_3=B_4=B_5=0$.

(III) *At least six of the A's and B's are positive* ($\lambda=4$, *throughout*).

Proof. By Lemma 3 not all the A's and B's can vanish. If all the A's are zero, (9.6) shows immediately that $B_1=B_2=B_3=B_4=B_5$. Likewise, if all the B's are zero, (9.6) shows that $A_1=A_2=A_3=A_4=A_5$. Hence, from now on we make the hypothesis H_1 *that not all the A's are zero and that not all the B's are zero.*

There are two possible cases: Either (Case I) two adjacent B's, say B_i and B_{i+1}, are both positive, or (Case II) there is no pair of adjacent positive B's. We consider these two cases separately.

Case I. We first prove the following subtheorem: *If* $B_i>0$ *and* $B_{i+1}>0$, *then either* $B_{i+2}>0$, *or six of the A's and B's are positive, or both alternatives hold.*

Proof of subtheorem. Without loss of generality we may take $i=3$, since subscripts may always be advanced modulo 5. Since $B_3>0$ and $B_4>0$, Lemma 1 shows that either $B_5>0$, in which case the subtheorem is true, or $A_1>0$. Thus, we may now assume B_3, B_4, A_1 all greater than zero. Lemma 3 shows also that at least one of the quantities A_3, B_1, B_5 is positive. If $B_5>0$, the subtheorem is true. If $B_1>0$, then by Lemma 1 we have B_2 or $A_3>0$ and B_5 or $A_4>0$, which, together with $B_3>0$, $B_4>0$, $A_1>0$, $B_1>0$, gives us six A's and B's greater than zero.

Hence we may limit attention to the case when B_3, B_4, A_1, A_3 are all positive. But since $A_3>0$, Lemma 2 shows that either $B_1>0$ or $A_5>0$.

If $B_1>0$, then either B_5 or A_4 is greater than 0 by Lemma 1 and six of the A's and B's are positive, namely B_3, B_4, A_1, A_3, B_1, and either B_5 or A_4.

If $A_5>0$, we so far have only five of the A's and B's which we know to be positive, namely B_3, B_4, A_1, A_3, A_5. Lemma 3, however, tells us that at least one of the quantities B_1, B_2, A_4 must also be positive and thus the proof of the subtheorem is complete.

We see, too, that under Case I (that is, $B_i>0$, $B_{i+1}>0$) we must always have six of the A's and B's greater than zero. For, otherwise, the above subtheorem successively shows $B_{i+2}>0$, $B_{i+3}>0$, $B_{i+4}>0$. Thus all five of the B's are positive and, by hypothesis H_1, not all the A's are zero. We thus arrive at a contradiction and Case I is completely disposed of.

Case II. To cover this case we introduce hypothesis H_2: *There are no adjacent B's, both of which are positive.* Notice that, under this hypothesis H_2, Lemma 1 no longer offers us a pair of alternatives. If $B_1>0$, as we may assume without loss of generality by hypothesis H_1, both A_3 and A_4 are positive. Also Lemma 2 shows us that A_1 must be positive. For the fact that $A_3>0$ means either that $B_5>0$ (which is excluded by H_2) or $A_1>0$. Thus we have B_1, A_1, A_3, A_4, all greater than 0. Moreover Lemma 3 shows that either B_2, B_3, or $A_5>0$. Lemma 3 also shows that either B_4, B_5, or $A_2>0$. Two of these possibilities (namely, $B_2>0, B_5>0$) are excluded by hypothesis H_2, but this is of no importance in reaching the now obvious conclusion that in Case II we always have at least six of the A's and B's greater than zero. This completes the proof of the theorem.

The 4c. reducibility of the proper 5-ring having more than a single region on each side is an immediate corollary of this theorem. In fact, if the map P of §7 were 4c. irreducible, $L_i^{\text{in}}(4)>0$ and $L_i^{\text{ex}}(4)>0$, $i=1, 2, \cdots, 5$, inasmuch as each of the ten maps referred to has fewer regions than P. If six of the A^{in}'s and B^{in}'s were positive, and if at least five of the A^{ex}'s and B^{ex}'s were positive (as they must be by Theorem I), formula (7.1) shows that $P(4)>0$. Hence the possibility (III) of Theorem I is ruled out for the map M^{in}. It is similarly ruled out for the map M^{ex}. It then becomes evident that $P(4)=0$ only if possibility (I) holds for one of the maps, M^{in} and M^{ex} (say M^{in}), and possibility (II) holds for the other. That is, if P is irreducible, we are thus led to (say) $B_1^{\text{ex}}=B_2^{\text{ex}}=B_3^{\text{ex}}=B_4^{\text{ex}}=B_5^{\text{ex}}=0$. But this would mean that M^{ex} could not be colored in four colors. This is a contradiction of our supposition that P was irreducible by reason of the fact that the ring surrounds more than one region in P so that P has more regions than M^{ex} or M^{in}. Hence P can not be 4c. irreducible.

12. **Further consequences of the analysis of the 5-ring.** It is possible to obtain numerous results analogous to those of §6. In the first place, we remark that the map K_1 may be regarded as more or less completely arbitrary. In fact, any map of triple vertices containing at least one region with three or more sides can be denoted by K_1. The other maps, M, $K_2, \cdots, K_4$, $L_1, \cdots, L_5$, are then obtained as follows: To get M, we select a region R of K_1 having three or more sides. Let l_1 and l_2 be two boundary lines of R abutting on a common vertex. We next draw a new boundary l_4 in R (see dotted line in figure 28) sufficiently near l_1 and l_2, so that the three lines

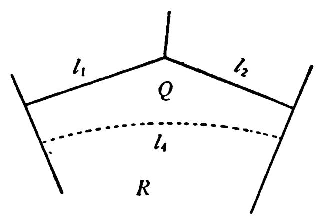


FIG. 28

l_1, l_2, l_4 are three sides of a new pentagon Q with l_1 and l_2 opposite to l_4. The maps $K_2, \cdots, K_5, L_1, \cdots, L_5$ are then obtained from M, as in §8, by omitting various boundaries of Q.

The chromatic polynomials associated with the maps so obtained will then be found to satisfy numerous inequalities which can be read off at once from (8.6). Not only are the left members of these identities not less than 0 for $\lambda=3, 4, 5, \cdots$, but they are completely monotonic for $\lambda \geqq 5$, a fact whose proof will be omitted on account of the close analogy with §6. Hence the right members of (8.6) are non-negative for $\lambda=3$ and 4 and are completely monotonic for $\lambda \geqq 5$.

Finally, we have the following curious result:

THEOREM I. *If K_1 is* 4c. *irreducible, then none of the other K's are. Moreover*

$$3K_3(4) = 6L_4(4) = 2L_5(4), \qquad 3K_4(4) = 2L_2(4) = 6L_3(4),$$
$$K_2(4) = L_1(4) + L_4(4), \qquad K_5(4) = L_1(4) + L_3(4).$$

Proof. Since $K_1(4)=0$, (8.2) shows that

$$A_1 = A_3 = A_4 = B_1 = 0. \tag{12.1}$$

Equalities (9.6) then become (for $\lambda=4$ and $i=1, 2, 3, 4$)

$$B_2 = B_3, \quad A_2 + B_3 = B_4 + A_5, \quad B_4 = B_5, \quad B_5 = A_2. \tag{12.2}$$

It follows from (12.1) and (12.2) that $B_2=B_3=A_5=\alpha$, say, and $B_4=B_5=A_2$ $=\beta$, say. Referring back to (8.2), we now have

$$(12.3)\qquad \begin{aligned} K_2 &= A_2 + A_4 + A_5 + B_2 = 2\alpha + \beta, \\ K_3 &= A_3 + A_5 + A_1 + B_3 = 2\alpha, \\ K_4 &= A_4 + A_1 + A_2 + B_4 = 2\beta, \\ K_5 &= A_5 + A_2 + A_3 + B_5 = \alpha + 2\beta; \end{aligned}$$

$$(12.4)\qquad \begin{aligned} L_1 &= A_1 + B_3 + B_4 = \alpha + \beta, \\ L_2 &= A_2 + B_4 + B_5 = 3\beta, \\ L_3 &= A_3 + B_5 + B_1 = \beta, \\ L_4 &= A_4 + B_1 + B_2 = \alpha, \\ L_5 &= A_5 + B_2 + B_3 = 3\alpha. \end{aligned}$$

Since K_1 is irreducible and L_i has one region less than K_1, it follows that $L_i>0$ for $i=1, 2, 3, 4, 5$. Hence from (12.4) we have $\alpha>0$ and $\beta>0$. This result together with (12.3) shows that $K_i(4)>0$ for $i=2, 3, 4, 5$. The rest of the theorem follows from the elimination of α and β from (12.3) and (12.4).

CHAPTER VI. PARTIAL ANALYSIS OF THE n-RING WITH SPECIAL ATTENTION TO THE 6-RING AND 7-RING

1. **The elementary maps and fundamental constrained polynomials entering into the theory of the n-ring.** The thoroughgoing analogy between the results obtained for the 4-ring and those obtained for the 5-ring indicates the possibility of formulating a general theory for the n-ring. Since a complete formulation of such a general theory has so far eluded us, it seemed desirable to make independent studies of the essentially simple 4-ring and 5-ring. Now, however, it appears well to introduce our studies of the much more complicated 6-ring and 7-ring by such general remarks as it is possible to make with regard to the n-ring.

Our theory really concerns a *class* C_n of *marked*(24) maps of triple vertices and simply connected regions. Each map M_n of the class C_n contains an n-gon marked Q_n surrounded by an n-ring whose regions are marked $R_1, R_2, \cdots, R_n$ in the cyclic order in which they occur. This cyclic order is supposed to be taken in the same sense for every map of the class. Two maps of class C_n are regarded as *essentially the same* if, and only if, they are topologically equivalent *and* the continuous one-to-one transformation which establishes this topological equivalence can be chosen in such a way that the regions marked Q_n and R_i in one map correspond respectively with the regions marked Q_n and R_i in the other map ($i=1, 2, \cdots, n$). If they are not essentially the same, they are *essentially distinct*.

The total number of regions in such a map M_n can be any integer greater

(24) The work "marked" has a sense here quite different from its particular meaning in Chapter III.

than $n+1$, but it can also be equal to $n+1$ or even less than $n+1$, inasmuch as the ring $R_1, \cdots, R_n$ need not be a proper ring, and, in particular, these n regions are not necessarily distinct. If the map M_n has no region (other than Q_n) which does not abut on Q_n, it will have at most $n+1$ distinct regions and will be called an *elementary* map of the class C_n. Let the number of essentially distinct proper elementary maps of the class C_n be denoted by $\mu(n)$. A glance at the previous chapter will show that $\mu(4)=4$, $\mu(5)=10$. It can also be verified without much trouble that $\mu(6)=34$. But we have not been able to deduce a formula for $\mu(n)$ [25].

Let us consider four elementary maps $M_n^{\rm I}$, $M_n^{\rm II}$, $M_u^{\rm III}$, $M_n^{\rm IV}$ satisfying the following five conditions:

(1) In $M_n^{\rm I}$, R_i has contact with R_k across a boundary with its end points abutting on R_j and R_l $(i<j<k<l\leqq n)$.

(2) In $M_n^{\rm II}$, R_i is identical with R_k.

(3) In $M_n^{\rm III}$, R_j has contact with R_l across a boundary with its end points abutting on R_i and R_k.

(4) In $M_n^{\rm IV}$, R_j is identical with R_l.

(5) All contacts and identifications of the R's other than those mentioned above are the same for each of the four maps.

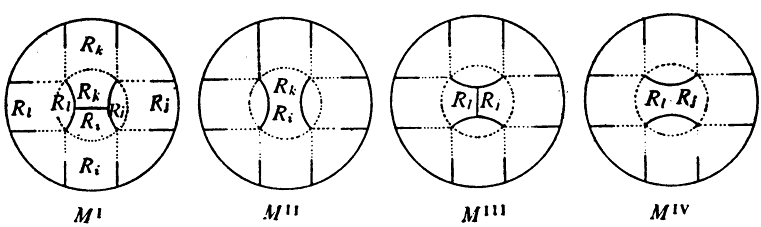


FIG. 29

Then it is clear by the Fundamental Principle (1.3) of Chapter I that the chromatic polynomials (either free or constrained) associated with the above four elementary maps must satisfy the identity

$$(1.1) \qquad M_n^{\rm I}(\lambda) + M_n^{\rm II}(\lambda) = M_n^{\rm III}(\lambda) + M_n^{\rm IV}(\lambda).$$

Thus any free or constrained chromatic polynomial for one of these maps is obtainable at once from the corresponding polynomials for the other three by the linear relation (1.1), in which the coefficients are independent of the nature of the constraints (so long as the R's are actually required to be colored). Hence four maps of this type will not be regarded as forming a set of mutually independent maps. In general, a set of elementary maps among whose constrained chromatic polynomials there are no linear relations (with

[25] The number of elementary maps with $n+1$ regions is, however, known since Euler to be $2^{n-2}\cdot 1\cdot 3\cdot 5 \cdots (2n-5)/(n-1)!$. Cf. Whitney [4, p. 211].

coefficients not all zero), which are independent of the nature of the constraints (so long as each R is required to be colored), will be said to form a "basic" set of mutually independent maps. A maximal basic set will be a basic set S such that the constrained polynomial of every elementary map not in S can be expressed linearly in terms of the corresponding polynomials for maps in S, the coefficients being independent of the nature of the constraints. Evidently, the number $\nu(n)$ of elementary maps in a maximal basic set is independent of the way in which the maximal set is chosen, inasmuch as $\nu(n)$ is merely the rank of a certain matrix with $\mu(n)$ rows and a number of columns equal to the number of types of constraint (in which each R is required to be colored).

It is clear from (1.1) that $\nu(n)$ must be less than $\mu(n)$ for $n \geqq 4$. Again, a glance at the preceding chapter shows that $\nu(4)=3$, $\nu(5)=6$, while it is also possible to show that $\nu(6)=15$, $\nu(7)=36$. For the complete theory of the n-ring it would be even more important to determine the function $\nu(n)$ than the function $\mu(n)$. It is hard to say which of these problems is more difficult.

Let us go back to the general marked map M_n (not necessarily an elementary map) of the class C_n. In any coloring of the ring, the regions $R_1, R_2, \cdots, R_n$ fall into a number of sets such that all regions of the same set have the same color, but any two regions from different sets have distinct colors. Two colorings of the ring belong to the same scheme if this division into sets is the same for the two colorings. For instance, if the six regions $R_1, \cdots, R_6$ of a six-ring are colored with the colors a, b, a, c, a, d respectively, we would have the same scheme as if they were colored b, c, b, a, b, d or x, y, x, z, x, w; but the coloring d, a, b, a, c, a would belong to a different scheme. Evidently a scheme is completely defined by the specification of one of its colorings. This will be our practice in the tables of §§2 and 5 below. A general formula for the number $\rho(n)$ of distinct schemes may be obtained as follows:

Let $F_n(\lambda)$ equal the number of ways of coloring an n-ring in λ colors. Then from (3.4) of Chapter II, we have $F_n(\lambda)=(\lambda-1)^n+(-1)^n(\lambda-1)$. Moreover $F_n(\lambda) = m_n\lambda(\lambda-1)\cdots(\lambda-n+1) + m_{n-1}\lambda(\lambda-1)\cdots(\lambda-n+2)+\cdots +m_2\lambda(\lambda-1)$, where m_k equals the number of schemes involving exactly k colors, so that

$$\rho(n) = \sum_{k=2}^{n} m_k, \qquad n \geqq 2. \tag{1.2}$$

However, we know from the calculus of finite differences that

$$m_k = \frac{\Delta^k F_n(0)}{k!}.$$

Also from the calculus of finite differences, we have for $k \geqq 2$

$$\Delta^k F_n(\lambda) = \Delta^k(\lambda - 1)^n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (\lambda - 1 + k - j)^n.$$

Hence

$$m_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(k - j - 1)^n}{k!}. \tag{1.3}$$

Combining (1.2) and (1.3), we have

$$\rho(n) = \sum_{k=2}^{n} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(k - j - 1)^n}{k!}.$$

This double sum may be written in the following slightly more convenient form:

$$\rho(n) = \frac{(n-1)^n}{n!} + \sum_{h=2}^{n-2} \frac{(h-1)^n}{h!} E(n-h) + (-1)^n E(n), \qquad n \geqq 4, \tag{1.4}$$

where

$$E(m) = \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^m}{m!}, \qquad m \geqq 2. \tag{1.5}$$

The first few of the E's are listed here for convenience: $E(2)=1/2$, $E(3)=1/3$, $E(4)=3/8$, $E(5)=11/30$, $E(6)=53/144$, $E(7)=103/280$, $E(8)=2119/5760$. Using these values for the E's, formula (1.4) yields $\rho(4)=4$, $\rho(5)=11$, $\rho(6)=41$, $\rho(7)=162$, $\rho(8)=715$. For computational purposes it is, however, much easier to find $m_0=0$, $m_1=0$, m_2, m_3, $\cdots$ as successive remainders upon dividing $F_n(\lambda)$ by λ, $\lambda-1$, $\lambda-2$, $\lambda-3$, $\cdots$ and then to find $\rho(n)$ from (1.2).

Let the various schemes be denoted by $A_1, A_2, \cdots, A_{\rho(n)}$: Let $A_i(\lambda)$ denote the number of ways $M_n - Q_n$ can be colored in λ colors so that the ring surrounding Q_n is colored according to the scheme A_i. The $A(\lambda)$'s are thus constrained chromatic polynomials; moreover they form a complete set of constrained polynomials in the sense that an arbitrary constrained polynomial $P(\lambda)$ for which the constraints are carried by no regions other than $Q_n, R_1, \cdots, R_n$(26), but with each R required to be colored, may be expressed in the form

$$P(\lambda) = \sum_{i=1}^{\rho(n)} R_i(\lambda) A_i(\lambda), \tag{1.6}$$

where the coefficients $R_i(\lambda)$ may be rational functions of λ but are independent of the particular M_n under consideration. For example, if $P(\lambda)$ is the free chromatic polynomial for M_n, then $R_i(\lambda) = \lambda - \alpha_i$, where α_i is the number of

(26) The free chromatic polynomial for M_n may be regarded as a special case of a constrained polynomial, for which the constraint-carrying regions form a vacuous set.

colors in the scheme A_i. Another obvious and important example is given by the following identity:

$$A_i^*(\lambda) = \frac{1}{\lambda(\lambda-1)\cdots(\lambda-\alpha_i+1)} A_i(\lambda), \qquad (\lambda \geqq \alpha_i), \tag{1.7}$$

where $A_i^*(\lambda)$ denotes the number of ways $M_n - Q_n$ can be colored so that a certain preassigned color (selected from the λ colors) is assigned to each region in the ring, the whole ring being colored in the scheme A_i. We omit the general proof of (1.6).

One of the primary objects is to discuss homogeneous linear identities of the form

$$\sum_{i=1}^{\rho(n)} a_i(\lambda) A_i(\lambda) = 0, \tag{1.8}$$

where the coefficients $a_i(\lambda)$ are the same for any map of the class C_n. In particular, (1.8) would have to be satisfied in the case of an elementary map of the class C_n. Let $M_n^{(1)}, M_n^{(2)}, \cdots, M_n^{(\mu(n))}$ denote the elementary maps of the class C_n and let $A_i^{(k)}(\lambda)$ denote the number of ways $M_n^{(k)} - Q_n$ can be colored in λ colors according to the scheme A_i ($k=1, 2, \cdots, \mu(n)$; $i=1, 2, \cdots, \rho(n)$). It follows that a system of admissible $a(\lambda)$'s must satisfy

$$\sum_{i=1}^{\rho(n)} a_i(\lambda) A_i^{(k)}(\lambda) = 0, \qquad k = 1, 2, \cdots, \mu(n). \tag{1.9}$$

These necessary conditions will next be shown to be sufficient. In fact, according to §6 of Chapter I, there exist identities of the form,

$$A_i(\lambda) = \sum_{k=1}^{\mu(n)} B_k(\lambda) A_i^{(k)}(\lambda), \qquad i = 1, 2, \cdots, \rho(n), \tag{1.10}$$

where $B_k(\lambda)$ depends upon the particular map M_n under consideration, but is independent of the nature of the constraints, that is, independent of i. Hence if (1.9) holds, we see from (1.10) that (1.8) must also hold. Hence in order to find all relations of the form (1.8) we have merely to find all solutions of equations (1.9), the a's being the unknowns and the $A_i^{(k)}(\lambda)$'s the known quantities[27].

The equations (1.9) are not linearly independent[28]. In fact the rank of the matrix is not less than $\nu(n)$ but on account of the fact that (1.6) holds for any constrained polynomial $P(\lambda)$, it follows that the rank of the matrix is exactly $\nu(n)$ and we can in fact limit attention to a maximal basic set of

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(27) The A's are very easily computed in any given case. It is, in fact, a priori evident that $A_i^*(\lambda)$ is either 0 or $\lambda(\lambda-1)\cdots(\lambda-\alpha_i+1)$ according as M^k can not be, or can be, colored in the scheme A_i.

(28) We are here assuming that $\mu(n) \leqq \rho(n)$, which is true for small values of n but has not been proved for all n.

elementary maps. If the notation has been chosen so that the first $\nu(n)$ of the M's form such a maximal basic set, we need only consider the first $\nu(n)$ of equations (1.9). In this way we see that *the number of linearly independent relations of the form* (1.8) *is equal to* $\rho(n)-\nu(n)$.

2. **The problem of expressing the constrained polynomials in terms of free polynomials.** In the application of the theory of the n-ring to the general theory of *free* chromatic polynomials it would appear to be important to express the $A(\lambda)$'s in terms of *free* chromatic polynomials. This is what was done in the previous chapter for $n=4$ and $n=5$ (cf. (2.3) and (8.6)). We now discuss the possibility of doing this in the general case.

For convenience let us think of the region Q_n of our marked map M_n as occupying a hemisphere, say the "northern" hemisphere, of a sphere, while the rest of the map occupies the "southern" hemisphere. Moreover we can assume that the vertices of Q_n lie at equal intervals along the equator.

Let $M_n^1, \cdots, M_n^{\nu(n)}$ be a maximal basic set of elementary maps for the class C_n of marked maps, while M_n, as before, represents a generic member of C_n, either elementary or not. Corresponding to each M_n we now define maps K_i $(i=1, 2, \cdots, \nu(n))$ as follows:

First reflect the configuration on the southern hemisphere of $M_n^{(i)}$ across the equator. In other words, move each point of the southern hemisphere of $M_n^{(i)}$ through the interior of the sphere parallel to the polar axis until it intersects the northern hemisphere. In this way we get a configuration $\overline{M}_n^{(i)}$ in the northern hemisphere of a sphere σ, while the southern hemisphere of σ is thought of as empty. Let $\overline{R}_1, \cdots, \overline{R}_n$ be the regions in the northern hemisphere of σ which correspond respectively to the regions $R_1, \cdots, R_n$ of the map $M_n^{(i)}$. *Second*, take the configuration on the southern hemisphere of the generic map M_n and place it on the empty southern hemisphere of σ, being careful to make the region R_i of the map M_n have contact at the equator with $\overline{R}_i$ but with no other of the $\overline{R}$'s $(i=1, 2, \cdots, n)$. *Third*, remove the boundary at the equator so that R_i and $\overline{R}_i$ (for each i) form one region. The resulting map on the sphere σ is the map K_i previously alluded to.

If $R_1, \cdots, R_n$ is not a proper ring in the map M_n, the map K_i may possibly turn out to be a pseudo-map or may have doubly connected regions. Otherwise K_i will be a proper map of triple vertices and simply connected regions. We use $K_i(\lambda)$ to denote the free chromatic polynomial associated with K_i. It is obvious that

$$K_i(\lambda) \equiv \sum_{j=1}^{\rho(n)} \epsilon_{ij} A_j(\lambda), \qquad i = 1, 2, \cdots, \nu(n), \tag{2.1}$$

where $\epsilon_{ij}=1$ or 0 according as the map $M_n^{(i)}$ can or can not be colored in the scheme A_j. We next would like to prove that the $\nu(n)$ identities (2.1) are linearly independent in the sense that there exist no multipliers $m_1(\lambda), \cdots, m_\nu(\lambda)$, not all zero, independent of M_n, such that $\sum_{i=1}^{\nu(n)} m_i(\lambda)K_i(\lambda)=0$. This is true for $n=4$ and $n=5$, but we have not been able to prove it in the general case. It must depend somehow on the fact that $M_n^1, \cdots, M_n^{\nu(n)}$ form a basic set. Indeed it is easy to see that, if the $M_n^{(i)}$ did not form a basic set in virtue of a situation like that described in connection with equality (1.1), the Fundamental Principle (1.3) of Chapter I would yield an identity of the type $K^{\mathrm{I}}(\lambda)+K^{\mathrm{II}}(\lambda)\equiv K^{\mathrm{III}}(\lambda)+K^{\mathrm{IV}}(\lambda)$ for the four associated K-maps. This gap in our theory is due to insufficient knowledge concerning the matrix (ϵ_{ij}) in the general case.

We next consider $\rho(n)-\nu(n)$, linearly independent, solutions of (1.9): $a_i=a_{ij}(\lambda)$; $i=1, \cdots, \rho(n)$; $j=1, \cdots, \rho(n)-\nu(n)$. Substituting in (1.8) we obtain

$$(2.2) \qquad \sum_{i=1}^{\rho(n)} a_{ij}(\lambda)A_i(\lambda) = 0, \qquad i = 1, 2, \cdots, \rho(n) - \nu(n).$$

These $\rho(n)-\nu(n)$ linearly independent equations taken with the $\nu(n)$ equations (2.1) give us just the right number of equations to solve for the $\rho(n)$ unknowns $A_1(\lambda), \cdots, A_{\rho(n)}(\lambda)$, provided, of course, that we are correct in believing that (2.1) and (2.2) yield $\rho(n)$ independent equations. Since, however, $\rho(n)$ is quite large for $n\geqq 6$, the practical difficulties in carrying out this solution appear almost insuperable.

3. General linear relationships for the fundamental constrained polynomials found by use of Kempe chains. As already explained, a complete set of linear relations of the form (1.8) can always be found by obtaining a complete solution of (1.9). The number $\nu(n)$ of independent equations is, however, fairly large for $n\geqq 6$ and hence this process is apt to be rather involved. We now explain how a large number of relations of the type (1.8) can be written down at once. These relations form a complete set for $n=4$, $n=5$, $n=6$, and $n=7$; but we have not proved that they form a complete set for $n\geqq 8$. Our results in this connection are summarized in the following theorem:

THEOREM I. *Let the regions of the n-ring* ($n\geqq 4$) *be divided into* $2k$ *sets,* $S_1, S_2, \cdots, S_{2k}$ ($2\leqq k\leqq n/2$), S_i *consisting of regions* $R_{p_i+1}, R_{p_i+2}, \cdots, R_{p_{i+1}}$, *where the p's are arbitrary integers satisfying* $0=p_1<p_2< \cdots <p_{2k+1}=n$.

Now consider any scheme, A_0, *say, for which the colors of the regions in the sets* $S_1, S_3, \cdots, S_{2k-1}$ *are distinct from the colors of* $S_2, S_4, \cdots, S_{2k}$. *Let* A_0^* *be a particular coloring belonging to the scheme* A_0, *the regions of* $S_1, S_3, \cdots, S_{2k-1}$ *being colored in some or all of the two or more colors* $a, b, \cdots$ *and the regions of* $S_2, S_4, \cdots, S_{2k}$ *being colored in some or all of the two or more colors* $c, d, \cdots$, *distinct from* $a, b, \cdots$. *Let* Π_1 *denote an arbitrary permutation of the colors* $a, b, \cdots$ *and* Π_2 *an arbitrary involutory permutation of* $c, d, \cdots$.

Let A_i^* ($i=1, 2, \cdots, 2^{k-2}-1$) *denote the coloring of the ring obtained from* A_0^* *by applying* Π_2 *to the colors of the regions in* S_{2l} *if, and only if, the lth digit (from the right) in the binary expansion of i is* 1 ($l=1, 2, \cdots, k-2$), *the colors of all other regions being left fixed. This definition is to be interpreted for* $k=2$ *in* *the sense that in this case the set of A^*'s to be defined is vacuous.*

Let B_i^* $(i=0, 1, 2, \cdots, 2^{k-2}-1)$ *denote the coloring of the ring obtained from* A_i^* *by applying* Π_1 *to* S_{2k-1} *and* Π_2 *to* S_{2k}. *Let* C_i^* *denote the coloring obtained from* A_i^* *by applying* Π_1 *to* S_{2k-1}. *Let* D_1^* *denote the coloring obtained from* A_i^* *by applying* Π_2 *to* S_{2k}. *Finally, let* A_i, B_i, C_i, D_i *respectively denote the schemes to which the colorings* $A_i^*, B_i^*, C_i^*, D_i^*$ *belong.*

Then (after all this elaborate definition of the 2^k *schemes) our theorem simply states that*

$$\sum_{i=0}^{2^{k-2}-1}\left(\frac{A_i(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\alpha_i+1)}+\frac{B_i(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\beta_i+1)}\right) \tag{3.1}$$
$$=\sum_{i=0}^{2^{k-2}-1}\left(\frac{C_i(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\gamma_i+1)}+\frac{D_i(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\delta_i+1)}\right),$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ *denote respectively the number of colors in the schemes* A_i, B_i, C_i, D_i.

Proof. We make the temporary assumption that λ is an integer not less than the number λ^* of colors, $a, b, \cdots, c, d, \cdots$ already mentioned in the statement of the theorem. Some of our λ assigned colors we identify with $a, b, \cdots, c, d, \cdots$ and, if $\lambda>\lambda^*$, there will be also some additional colors, $e, f, \cdots, g, h, \cdots$. These λ colors are now divided into two complementary sets, ϕ and ψ, where the set ϕ includes the colors $a, b, \cdots$ and perhaps some additional colors, $e, f, \cdots$ and ψ includes $c, d, \cdots$ and perhaps some additional colors $g, h, \cdots$ [29]. We next let $A_i^*(\psi)$, $B_i^*(\psi)$, $C_i^*(\psi)$, $D_i^*(\psi)$ denote respectively the number of ways of coloring M_n-Q_n so that the colorings $A_i^*, B_i^*, C_i^*, D_i^*$ appear in the ring and so that the regions of S_{2k-2} and of S_{2k} are connected by a chain of regions (a ψ-chain) colored in the colors of ψ. We let $A_i^*(\phi, l)$, $B_i^*(\phi, l)$, $C_i^*(\phi, l)$, $D_i^*(\phi, l)$ denote respectively the number of ways of coloring M_n-Q_n so that the colorings $A_i^*, B_i^*, C_i^*, D_i^*$ appear in the ring and so that the regions S_{2k-1} are connected with the regions S_{2l+1} $(l=0, 1, \cdots, k-2)$ by a chain of regions (a ϕ-chain) colored in the colors of ϕ, but so that S_{2k-1} is not connected with S_{2h+1} for $h<l$ by a ϕ-chain. Finally we let $A_i^*(\lambda)$, $B_i^*(\lambda)$, $C_i^*(\lambda)$, $D_i^*(\lambda)$ denote respectively the total number of ways of coloring M_n-Q_n so that the colorings $A_i^*, B_i^*, C_i^*, D_i^*$ appear in the ring. Then it is evident that

$$A_i^*(\lambda)=A_i^*(\psi)+\sum_{l=0}^{k-2}A_i^*(\phi, l),\qquad B_i^*(\lambda)=B_i^*(\psi)+\sum_{l}B_i^*(\phi, l), \tag{3.2}$$
$$C_i^*(\lambda)=C_i^*(\psi)+\sum_{l}C_i^*(\phi, l),\qquad D_i^*(\lambda)=D_i^*(\psi)+\sum_{l}D_i^*(\phi, l),$$

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[29] As far as the needs of the proof are concerned, all the additional colors could be lumped into one or the other of the sets ϕ or ψ. We have chosen the more general viewpoint for aesthetic reasons.

since, in any coloring of $M_n - Q_n$ of the types considered, either S_{2k-2} is connected with S_{2k} by a ψ-chain or S_{2k-1} is connected with at least one of the sets $S_1, S_3, \cdots, S_{2k-3}$ by a ϕ-chain. Moreover, in the former case we can perform an arbitrary permutation of the colors of ϕ on one side of the ψ-chain while holding all colors fast on the other side. In the latter case we can perform a permutation of the colors of ψ on one side of the ϕ-chain while holding all colors fast on the other side. From these considerations a detailed study shows that

$$(3.3)\qquad \begin{aligned} A_i^*(\psi) &= C_i^*(\psi), \qquad i = 0, 1, 2, \cdots, 2^{k-2} - 1, \\ B_i^*(\psi) &= D_i^*(\psi); \end{aligned}$$

$$(3.4)\qquad \begin{aligned} A_i^*(\phi, l) &= D_j^*(\phi, l), \\ B_i^*(\phi, l) &= C_j^*(\phi, l), \qquad l = 0, 1, 2, \cdots, k - 2, \end{aligned}$$

where j is obtained from i by interchanging 0 and 1 in the last l digits of the binary expansion of i. In this connection it is assumed that both i and j have just exactly $k-2$ digits by prefixing, if necessary, a sufficient number of 0's. It is clear that as i ranges over all values from 0 to $2^{k-2}-1$ (with l fixed), the same is true of j. Hence, on summing (3.4) with respect to i, we get

$$(3.5)\qquad \sum_{i=0}^{2^{k-2}-1} A_i^*(\phi, l) = \sum_{i=0}^{2^{k-2}-1} D_i^*(\phi, l), \qquad \sum_i B_i^*(\phi, l) = \sum_i C_i^*(\phi, l).$$

Summing (3.5) with respect to l and (3.3) with respect to i and combining the four results by addition, we get

$$\sum_{i=0}^{2^{k-2}-1} \left[A_i^*(\psi) + \sum_{l=0}^{k-2} A_i^*(\phi, l) + B_i^*(\psi) + \sum_l B_i^*(\phi, l) \right]$$
$$= \sum_i \left[C_i^*(\psi) + \sum_l C_i^*(\phi, l) + D_i^*(\psi) + \sum_l D_i^*(\phi, l) \right].$$

This last equation is now simplified with the help of (3.2). We thus have

$$(3.6)\qquad \sum_i [A_i^*(\lambda) + B_i^*(\lambda)] = \sum_i [C_i^*(\lambda) + D_i^*(\lambda)].$$

Finally we eliminate the A^*'s, B^*'s, C^*'s and D^*'s from (3.6) by relations of the type (1.7) and thus obtain (3.1).

We have now proved (3.1) for integral values of $\lambda \geqq \lambda^*$. But since (3.1), after being cleared of fractions, is an equality between two polynomials, it follows immediately that (3.1) must hold identically, so that the theorem has finally been completely proved.

This theorem takes care of the cases $n=4$ and $n=5$ completely, and very cheaply, by taking $k=2$. Special cases of (3.1) with $k=2$ have occurred several times in the preceding chapter (cf. (2.2), (8.4), and (9.6) of Chapter V). To get a complete set of linear relations for the cases $n=6$ and $n=7$ we need to use (3.1) with $k=3$ and again with $k=2$.

There are undoubtedly still other theorems like Theorem I, but whether they would actually give linear relations independent of the ones given by Theorem I is not known. The number of relations that can be written down by this one theorem is enormous. This is due to the arbitrariness in the choice of k, in the choice of the sets $S_1, S_2, \cdots, S_{2k}$, in the choice of A_0, and finally in the choice of Π_1 and Π_2. In this connection it is not without interest to remark that for $k=2$, Π_2 (as well as Π_1) need not be involutory. This is of no significance if λ is not greater than 4.

4. **Linear inequalities.** The methods of the previous section also lead to certain results, which apparently can not be obtained by the methods of §§1–2. Our theory must be regarded as incomplete until it is shown how these results fit in with the considerations of these earlier sections.

Taking $i=0$ in (3.2) we write

$$A_0^*(\lambda) = A_0^*(\psi) + \sum_{l=0}^{k-2} A_0^*(\phi, l), \tag{4.1}$$

while from (3.3) we obtain

$$A_0^*(\psi) = C_0^*(\psi) \leqq C_0^*(\lambda), \tag{4.2}$$

and from (3.4) we obtain

$$A_0^*(\phi, l) = D_{2l-1}^*(\phi, l) \leqq D_{2l-1}^*(\lambda).$$

Summing this last inequality with respect to l, we get

$$\sum_{l=0}^{k-2} A_0^*(\phi, l) \leqq \sum_{l=0}^{k-2} D_{2l-1}^*(\lambda). \tag{4.3}$$

Combining (4.1), (4.2), and (4.3) we find that

$$A_0^*(\lambda) \leqq C_0^*(\lambda) + \sum_{l=0}^{k-2} D_{2l-1}^*(\lambda), \tag{4.4}$$

which can also be written in the following equivalent form:

$$\frac{A_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\alpha_0+1)} \leqq \frac{C_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\gamma_0+1)} + \sum_{l=0}^{k-2} \frac{D_{j(l)}(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\delta_{j(l)}+1)} \tag{4.5}$$

where $j(l)=2^l-1$.

It is to be noted that (4.5), unlike (3.1), has been established only for positive integral values of λ. Furthermore, for $k=2$, (4.5) yields no information not already contained in (3.1), inasmuch as the equality

$$\frac{A_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\alpha_0+1)}+\frac{B_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\beta_0+1)}$$
$$=\frac{C_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\gamma_0+1)}+\frac{D_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\delta_0+1)}$$

(in which the terms on both sides of the equality are non-negative) clearly implies that

$$\frac{A_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\alpha_0+1)}\leqq\frac{C_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\gamma_0+1)}$$
$$+\frac{D_0(\lambda)}{\lambda(\lambda-1)\cdots(\lambda-\delta_0+1)}.$$

But for $k=3$, (4.5) contains information that is essentially new.

5. **Fundamental linear relations for the six-ring.** In applying the results of the preceding sections to the 6-ring, we define the various schemes for the 6-ring by means of the following table:

	R_i	R_{i+1}	R_{i+2}	R_{i+3}	R_{i+4}	R_{i+5}
A	a	b	a	b	a	b
B_i	c	a	b	a	b	a
C_i	b	a	c	b	c	a
D	a	b	c	a	b	c
E_i	b	a	c	a	d	a
F_i	b	a	c	b	d	a
G_i	d	a	b	a	b	c
H_i	c	a	b	d	b	a
I_i	c	a	b	d	a	b
J_i	a	b	c	d	e	b
K_i	b	a	c	b	d	e
L	a	b	c	d	e	f

Here the letters $a, b, \cdots, f$ are used to indicate distinct colors. The integral index i is to be taken modulo 6. It is easily seen that $C_i = C_{i+3}$, $E_i = E_{i+2}$, $H_i = H_{i+3}$, $I_i = I_{i+3}$, $K_i = K_{i+3}$. Hence there is one A, there are six B's, three C's, one D, two E's, six F's, six G's, three H's, three I's, six J's, three K's, and one L, making a total of 41 distinct schemes in accordance with the value of $\rho(6)$ given in §1.

In accordance with the theorem of §3, it is possible to write down the following relations:

$$(\lambda - 3)[(\lambda - 2)A - B_1 - B_2 + B_4 - B_6] + G_1 + G_2 - H_1 = 0, \tag{5.1}$$

$$(\lambda - 3)[B_1 - C_1] - E_1 + F_1 = 0, \tag{5.2}$$

$$F_2 + F_3 - G_6 - I_1 = 0, \tag{5.3}$$

$$(\lambda - 3)[D - C_1] + H_1 - I_1 = 0, \tag{5.4}$$

$$(\lambda - 3)(\lambda - 4)B_1 + (\lambda - 4)[- E_1 - H_1] + J_1 = 0, \tag{5.5}$$

$$(\lambda - 4)[- F_1 + G_4] - J_6 + K_1 = 0, \tag{5.6}$$

$$(\lambda - 3)(\lambda - 4)(\lambda - 5)C_1 - (\lambda - 4)(\lambda - 5)H_1 - (\lambda - 5)K_1 + L = 0. \tag{5.7}$$

Other relations may be found from these seven by advancing the subscripts modulo 6. It is left to the reader to show that in this way it is possible to obtain exactly 26 linearly independent relations.

Equations (5.1) to (5.6) can be solved for A, the F's, G's, H's, J's, K's, and L in terms of the other fifteen quantities, B's, C's, D, E's, I's. We then get

$$\begin{aligned}
u(u+1)A &= u\left[\sum_{i=1}^{6} B_i - \sum_{i=1}^{3} C_i - D\right] - 2[E_1 + E_2] + \sum_{i=1}^{3} I_i, \\
F_i &= u[- B_i + C_i] + E_i, \\
G_i &= u[- B_{i+2} - B_{i+3} + C_i + C_{i+2}] + E_1 + E_2 - I_{i+1}, \\
H_i &= u[C_i - D] + I_i, \\
J_i &= u(u-1)[- B_i + C_i - D] + (u-1)[E_i + I_i], \\
K_i &= u(u-1)[- D] + (u-1)[I_{i+1} + I_{i-1}], \\
L &= u(u-1)(u-2)[- 2D] + (u-1)(u-2)\left[\sum_{i=1}^{3} I_i\right],
\end{aligned} \tag{5.8}$$

where $u = \lambda - 3$.

In order to express our fundamental constrained polynomials in terms of free polynomials, we proceed to form twelve maps, Z_1, Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, X_1, X_2, X_3, W_1, W_2 by the erasure of certain boundaries of Q as indicated in the following table:

Map	Boundaries of Q to be erased
Z_1	l_1
Y_1	l_{i-1}, l_{i+1}
X_i	l_i, l_{i+3}
W_i	$l_{i+1}, l_{i-1}, l_{i+3}$

Here l_i denotes the boundary between R_i and Q. We must also form three other maps V_1, V_2, V_3. To form V_i we divide Q into two regions by drawing a new boundary joining the vertex (Q, R_{i+1}, R_{i+2}) with the vertex (Q, R_{i-1}, R_{i-2}) and then erasing $l_{i+1}, l_{i+2}, l_{i-1}, l_{i-2}$.

By drawing figures representing these fifteen maps the reader can verify quite readily that the free chromatic polynomials for these maps may be expressed in terms of the forty-one constrained polynomials, $A, B_1, B_2, \cdots, L$ as follows:

$$
\begin{aligned}
Z_1 &= B_1 + E_1 + F_3 + F_5 + G_1 + G_2 + H_1 + I_1 \\
&\quad + J_1 + J_3 + J_4 + J_5 + K_2 + K_3 + L, \\
Y_i &= B_{i+3} + C_i + F_i + G_{i+3} + G_{i+4} + H_i + J_i, \\
X_i &= C_i + D + F_i + F_{i+3} + I_{i+1} + I_{i-1} + K_i, \\
W_i &= A + B_i + B_{i+2} + B_{i-2} + E_i, \\
V_i &= A + B_i + B_{i+3} + C_i + H_i.
\end{aligned}
\tag{5.9}
$$

These are equations (2.1) specialized for the case $n=6$. If in (5.9) we insert the values of A, F_i, G_i, H_i, J_i, K_i and L as given by (5.8), we obtain fifteen linear equations which could presumably be solved for the fifteen unknowns (six B's, three C's, one D, two E's, three I's) in terms of Z_1, Y_i, X_i, W_i, V_i. We have not carried through this solution on account of the large amount of work involved in the evaluation of fifteenth order determinants whose elements are polynomials in λ. If there is some essential short cut to the solution, we have not discovered it.

6. **The four-color reducibility of four pentagons surrounding a boundary.** It is well known (Birkhoff [1]) that any map which contains a boundary line separating two pentagons and having its end points on two other pentagons is 4c. reducible. We shall show that this result does not depend essentially upon chain theory. Nevertheless chain theory gives much more complete information about certain questions which arise in connection with this configuration.

We may evidently assume that our map M contains no proper rings of fewer than six regions except for the five-rings surrounding pentagons. The configuration consisting of the four pentagons will then be surrounded by a proper 6-ring (R_1, R_2, R_3, R_4, R_5, R_6). Let the subscripts be chosen so that R_1 has a boundary in common with just one of the four "interior" pentagons. The same will then also be true of R_4, but each of the other regions in the 6-ring will touch two pentagons. The first thing to be noted is that the ring and its interior can be colored in each of the following schemes for the ring (cf. §5 for notation): B_2, B_3, B_5, B_6, C_1, C_2, C_3, F_1, F_4, G_1, G_2, G_4, G_5, H_1, H_2, H_3([30]). In accordance with the previous notation, we also use A, B_1, B_2, and so on, to denote the number of ways the ring and its *exterior* can be colored in $\lambda=4$ colors in the corresponding schemes A, B_1, B_2, and so on. Hence if M can *not* be colored in 4 colors, we must have

$$(6.1)\qquad \begin{aligned} B_2 = B_3 = B_5 = B_6 = C_1 = C_2 = C_3 = F_1 = F_4 = G_1 = G_2 \\ = G_4 = G_5 = H_1 = H_2 = H_3 = 0. \end{aligned}$$

From these values for sixteen of our fundamental constrained polynomials (for $\lambda=4$), we see what can be deduced with the help of (5.1), (5.2), (5.3), and (5.4).

From $B_1+F_1=C_1+E_1$ (cf. (5.2)), we obtain $B_1=E_1$. From $B_2+F_2=C_2+E_2$, we get $F_2=E_2$. Our calculation can be reduced by considerations of symmetry, which yield the following: $B_4=E_4=E_2$, $F_3=E_3=E_1$, $F_5=E_5=E_1$, $F_6=E_6=E_2$. All these results may be summarized as follows:

$$(6.2)\qquad \begin{aligned} B_1 = E_1 = F_3 = F_5 = \alpha, \quad \text{say;} \\ B_4 = E_2 = F_2 = F_6 = \beta, \quad \text{say.} \end{aligned}$$

From $F_2+F_3=G_6+I_1$ (cf. (5.3)) we obtain $\alpha+\beta=G_6+I_1$ and (by symmetry) $\alpha+\beta=G_3+I_1$. From $F_3+F_4=G_1+I_2$ we obtain $\alpha=I_2$ and (by symmetry) $\beta=I_3=I_2$, $\alpha=I_3$. This shows among other things that $\alpha=\beta$, so that (6.2) can be revised as follows:

$$(6.3)\qquad \begin{aligned} B_1 = B_4 = E_1 = E_2 = F_2 = F_3 = F_5 = F_6 = I_2 = I_3 \\ = (G_3 + I_1)/2 = (G_6 + I_1)/2 = \alpha. \end{aligned}$$

This result combined with (5.1) and (6.1) shows immediately that

$$(6.4)\qquad A = 0.$$

But we can not, by this method, prove that α also vanishes. The best we can do, using (5.4), is to show that, in addition to (6.3), we can write $I_1=G_3=G_6=\alpha$. This will also satisfy all the equations obtained from (5.1) by advancing the subscripts. Using, however, an inequality of the type indicated by (4.5), for example,

$$(6.5)\qquad 0 \leqq B_1 \leqq B_5 + H_1 + G_2,$$

([30]) On account of the symmetry of the configuration it is only necessary to test the stated fact for the schemes B_2, C_1, C_2, F_1, G_1, H_1, H_2. It can also be verified that the ring and its interior can *not* be colored in any scheme other than the sixteen schemes listed above.

we see immediately from (6.1) that $\alpha = B_1$ must vanish. Hence, also, by (6.3), all the other quantities B_4, E_1, E_2, and so on, must vanish. This is obviously quite sufficient to show the four-color reducibility of our configuration. The original proof of Birkhoff likewise uses implicitly an inequality like (6.5) and hence is connected with the rather strong statement that $\alpha = 0$. *Nevertheless the 4c. reducibility of our configuration can also be proved from the weaker result embodied in* (6.4), which, since it does not depend upon inequalities like (6.5), is not essentially dependent on the theory of the Kempe chains. To prove the italicized statement the reader may verify that the number of ways of coloring the map V_2 (cf. §5) in 4 colors is equal to

$$A + B_2 + B_5 + C_2 + H_2$$

(cf. last of equations (5.9) with $i = 2$), which by (6.1) and (6.4) is equal to zero. The 4c. reducibility follows at once from the fact that the map V_2 has fewer regions (namely, 6 fewer) than M.

7. **Further consequences of the partial analysis of the six-ring.** We now consider some miscellaneous results concerning a 4c. irreducible map M which contains a hexagon Q surrounded by a (necessarily proper) 6-ring (R_1, R_2, R_3, R_4, R_5, R_6). The map Z_i is formed by omitting the boundary l_i between R_i and Q. For $i = 1$, this agrees with the previous definition of Z_1, and evidently the first of equations (5.9) is valid under cyclical advancement of the subscripts. In any coloring of $M - Q$ in four colors, it is clear that all four colors must appear in the 6-ring about Q; otherwise the whole map M could be colored. Hence, in this irreducible case, we have $A = B_i = C_i = D = 0$. Since we are now dealing exclusively with $\lambda = 4$, we also have $J_i = K_i = L = 0$. Hence equations (5.9) reduce to

$$\begin{aligned} Z_i &= E_i + F_{i+2} + F_{i+4} + G_i + G_{i+1} + H_i + I_i, \\ Y_i &= F_i + G_{i+3} + G_{i+4} + H_i, \\ X_i &= F_i + F_{i+3} + I_{i+1} + I_{i-1}, \\ W_i &= E_i, \\ V_i &= H_i. \end{aligned} \tag{7.1}$$

Likewise equations (5.8) reduce to

$$\begin{aligned} 2(E_1 + E_2) &= I_1 + I_2 + I_3, \\ F_i &= E_i, \\ G_i &= (E_1 + E_2) - I_{i+1}, \\ H_i &= I_i. \end{aligned} \tag{7.2}$$

Remembering that $E_{i+2} = E_i$ and that $H_{i+3} = H_i$, we can easily eliminate the E_i, F_i, G_i, H_i, and I_i between (7.1) and (7.2); the results are

$$
(7.3)\quad
\begin{aligned}
&Z_i = 3(W_i + V_i), && \sum_{i=1}^{6} Z_i = 21(W_1 + W_2) = \frac{21}{2}(V_1 + V_2 + V_3),\\
&Y_i = W_i + 2V_i, && \sum_{i=1}^{6} Y_i = 11(W_1 + W_2) = \frac{11}{2}(V_1 + V_2 + V_3),\\
&X_i = 3(W_1 + W_2) - V_i, && \\
& && X_1 + X_2 + X_3 = 7(W_1 + W_2) = \frac{7}{2}(V_1 + V_2 + V_3).
\end{aligned}
$$

Noting also that $W_{i+2} = W_i$ and $V_{i+3} = V_i$, we can obtain the following curious relations, some of which are special cases of theorems proved in the previous chapter:

$$
\begin{gathered}
Z_i + Z_{i+1} = Z_{i+3} + Z_{i+4}, \qquad Y_i + Y_{i+1} = Y_{i+3} + Y_{i+4},\\
2(Z_3 - Z_1) = 3(Y_3 - Y_1) = 6(X_1 - X_3),\\
Z_1 - Z_4 = 3(Y_1 - Y_4).
\end{gathered}
$$

8. **Fundamental linear relations for the 7-ring.** We define the various schemes for the 7-ring by means of the following table:

	R_i	R_{i+1}	R_{i+2}	R_{i+3}	R_{i+4}	R_{i+5}	R_{i+6}	No. of colors in ring
A_i	a	b	c	b	c	b	c	3
B_i	a	c	a	b	c	a	b	3
C_i	a	b	a	b	c	a	c	3
D_i	a	b	a	c	d	a	b	4
E_i	a	b	a	b	c	a	d	4
F_i	a	d	a	c	b	a	b	4
G_i	a	b	a	d	b	a	c	4
H_i	a	c	a	b	d	a	b	4
I_i	a	b	c	b	d	c	d	4
J_i	a	b	c	d	b	c	d	4
K_i	a	b	c	d	c	b	d	4
L_i	a	b	c	d	c	d	b	4
M_i	a	b	c	d	b	d	c	4

	R_i	R_{i+1}	R_{i+2}	R_{i+3}	R_{i+4}	R_{i+5}	R_{i+6}	No. of colors in ring
N_i	a	b	a	c	d	a	e	5
O_i	c	a	b	d	e	b	a	5
P_i	d	e	a	b	a	b	c	5
Q_i	c	a	d	a	b	e	b	5
R_i([31])	b	a	c	b	d	e	a	5
S_i	b	a	c	d	b	e	a	5
T_i	c	a	b	d	e	a	b	5
U_i	c	a	d	b	a	e	b	5
V_i	b	a	c	d	e	f	a	6
W_i	c	b	a	e	f	a	d	6
X	a	b	c	d	e	f	g	7

The integral index i is to be taken modulo 7. Since 7 is a prime, it is evident that there are seven A's, seven B's, $\cdots$, seven W's, and one X, making a total of 162 distinct schemes in accordance with the value of $\rho(7)$ given in §1. Using the Theorem of §3, we write down the following:

$$\begin{aligned} uA_1 + F_6 + K_3 + I_5 &= uA_2 + D_4 + L_3 + J_5 \qquad (u = \lambda - 3), \\ uA_1 + E_3 + M_6 + I_4 &= uA_7 + D_5 + L_6 + J_4, \\ uB_1 + G_2 + L_3 + E_7 &= uA_5 + J_1 + D_7 + L_6, \\ uB_1 + H_7 + L_6 + F_2 &= uA_4 + J_1 + D_2 + L_3, \\ uC_1 + G_2 + D_7 + K_6 &= uA_5 + I_1 + E_7 + M_3, \\ uC_1 + H_7 + D_2 + M_3 &= uA_4 + I_1 + F_2 + K_6; \end{aligned} \tag{8.1}$$

$$\begin{aligned} uB_1 + H_3 &= uB_3 + G_1, \\ uC_1 + F_4 &= uC_4 + E_1, \\ E_1 + K_4 &= G_1 + I_5, \\ F_1 + M_5 &= H_1 + I_4, \\ K_1 + J_7 &= M_7 + J_1; \end{aligned} \tag{8.2}$$

([31]) There is no possibility of confusing the scheme R_i with the region R_i.

$$
(8.3)\quad
\begin{aligned}
N_1 + (\lambda - 4)(\lambda - 3)B_1 &= (\lambda - 4)[G_1 + H_1],\\
N_1 + (\lambda - 4)I_4 &= (\lambda - 4)F_1 + S_2,\\
N_1 + (\lambda - 4)I_5 &= (\lambda - 4)E_1 + R_7,\\
O_1 + (\lambda - 4)I_3 &= (\lambda - 4)M_4 + Q_3,\\
O_1 + (\lambda - 4)I_6 &= (\lambda - 4)K_5 + Q_6,\\
P_1 + (\lambda - 4)(\lambda - 3)C_3 &= (\lambda - 4)[E_3 + L_1],\\
P_1 + (\lambda - 4)(\lambda - 3)C_6 &= (\lambda - 4)[F_6 + L_1],\\
T_1 + (\lambda - 4)K_5 &= (\lambda - 4)J_5 + O_1,\\
T_1 + (\lambda - 4)M_4 &= (\lambda - 4)J_4 + O_1,\\
U_1 + (\lambda - 4)I_1 &= Q_1 + (\lambda - 4)J_1,\\
U_1 + (\lambda - 4)J_2 &= T_6 + (\lambda - 4)J_1,\\
U_1 + (\lambda - 4)J_7 &= T_3 + (\lambda - 4)J_1;
\end{aligned}
$$

$$
(8.4)\quad
\begin{aligned}
V_1 + (\lambda - 5)(\lambda - 4)K_5 &= (\lambda - 5)[R_1 + O_1],\\
V_1 + (\lambda - 5)(\lambda - 4)M_4 &= (\lambda - 5)[S_1 + O_1],\\
W_1 + (\lambda - 5)(\lambda - 4)K_5 &= (\lambda - 5)[U_7 + O_1],\\
W_1 + (\lambda - 5)(\lambda - 4)M_4 &= (\lambda - 5)[U_2 + O_1];
\end{aligned}
$$

$$
(8.5)\quad
\begin{aligned}
X + (\lambda - 6)(\lambda - 5)(\lambda - 4)M_4 &= (\lambda - 6)(\lambda - 5)O_1 + (\lambda - 6)W_3,\\
X + (\lambda - 6)(\lambda - 5)(\lambda - 4)K_5 &= (\lambda - 6)(\lambda - 5)O_1 + (\lambda - 6)W_6.
\end{aligned}
$$

In connection with these equations it is to be emphasized that the subscripts may be advanced modulo 7. We have thus exhibited a total of 203 relations of the type indicated in (3.1) for the case of the 7-ring. The number of *independent* relations is, however, theoretically only 126. No attempt has been made to investigate the presence of 126 linearly independent relations among the 203 written down above. It is possible that our list of relations should be lengthened still more. It appears, however, from our subsequent investigation of a reducibility theorem due to Franklin that, so far as $\lambda=4$ is concerned, our list is virtually complete.

Our 162 fundamental constrained polynomials can presumably (after a stupendous algebraic calculation) be expressed in terms of the free polynomials for the 36 maps $\mathfrak{P}_i$, $\mathfrak{Q}_i$, $\mathfrak{R}_i$, $\mathfrak{S}_i$, $\mathfrak{T}_i$, and $\mathfrak{U}_i$ ($i=1, 2, \cdots, 7$) characterized as follows:

In $\mathfrak{P}_i$ the regions R_i, R_{i+2}, R_{i-2} are identical; that is, they are connected interior to the (necessarily improper) 7-ring.

In $\mathfrak{Q}_i$, R_{i+1} and R_{i-1} are identical, and R_{i+2} and R_{i-2} are identical.

In $\mathfrak{R}_i$, R_{i+1} and R_{i+3} are identical, and R_{i-1} and R_{i-3} are identical.

In $\mathfrak{S}_i$, R_{i+1} and R_{i-1} are identical and touch R_{i+3} and R_{i-3}.
In $\mathfrak{T}_i$, R_{i+2} and R_{i-2} are identical and touch R_i.
In $\mathfrak{U}_i$, R_i touches $R_{i\pm 2}$ and $R_{i\pm 3}$.

Before carrying out the computation referred to above, it would, of course, first be necessary to express $\mathfrak{P}_i(\lambda)$, $\mathfrak{Q}_i(\lambda)$, $\cdots$ in terms of $A_i(\lambda)$, $B_i(\lambda)$, $\cdots$. In illustration of these relations we give only the following, which we shall use in the next section:

$$(8.6) \quad \mathfrak{Q}_1 = A_5 + A_4 + B_7 + B_2 + D_1 + G_2 + H_7 + K_5 + M_4 + O_1.$$

9. **The four-color reducibility of three pentagons touching a boundary of a hexagon.** It is well known (Franklin [1]) that any map which contains a boundary line separating a hexagon and a pentagon and having its end points on two other pentagons is 4c. reducible. We shall give a proof of this theorem, which, depending upon equalities like (3.1) rather than inequalities like (4.5), is essentially independent of the Kempe chain theory.

We may evidently assume that our map M contains no proper rings of fewer than six regions except for the 5-rings surrounding pentagons. The configuration consisting of the hexagon and three pentagons will then be surrounded by a proper 7-ring $(R_1, R_2, \cdots, R_7)$. Let the subscripts be chosen so that R_1 touches the hexagon but does not touch any of the three pentagons. Using the notation of §8, it can be verified that the ring and its *interior* can be colored in each of the following schemes for the ring: A_4, A_5, C_1, C_2, C_3, C_6, C_7, D_4, D_5, E_1, E_2, F_1, F_7, G_2, G_5, H_4, H_7, I_1, I_3, I_4, I_5, I_6, L_i $(i=1, 2, 3, 4, 5, 6, 7)$, K_1, K_2, K_3, K_5, K_7, M_1, M_2, M_4, M_6, M_7(32). In accordance with the previous notation, we also use A_i, B_i, and so on, to denote the number of ways the ring and its *exterior* can be colored in $\lambda=4$ colors in the corresponding schemes A_i, B_i, and so on. Hence, if the map M can *not* be colored in 4 colors, we must have

$$(9.1) \quad \begin{aligned} A_4 &= A_5 = C_1 = C_2 = C_3 = C_6 = C_7 = D_4 = D_5 = E_1 = E_2 \\ &= F_1 = F_7 = G_2 = G_5 = H_4 = H_7 = I_1 = I_3 = I_4 = I_5 = I_6 \\ &= L_i = K_1 = K_2 = K_3 = K_5 = K_7 = M_1 = M_2 = M_4 \\ &= M_6 = M_7 = 0. \end{aligned}$$

A detailed calculation based on (8.1), (8.2) and (9.1) shows that $B_1=C_4 = C_5 = E_3 = E_4 = E_5=F_4=F_5=F_6=G_1=G_3=G_4=H_1=H_5=H_6=I_2=I_7=J_i =K_4=K_6=M_3=M_5$ but that the common value of these quantities is not necessarily zero. It is also found simultaneously that all the other quantities are zero. That is, we may now write

(32) On account of the symmetry of the configuration it is only necessary to test the stated fact for the schemes A_4, C_1, C_2, C_3, D_4, E_1, E_2, G_2, G_5, I_1, I_3, I_4, L_1, L_2, L_3, L_4, K_1, K_2, K_3, K_5, K_7. It can also be verified that the ring and its interior can *not* be colored in any scheme other than the 39 schemes listed above.

$$
\begin{aligned}
A_i &= B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = C_1 = C_2 = C_3 = C_6 = C_7 \\
&= D_i = E_1 = E_2 = E_6 = E_7 = F_1 = F_2 = F_3 = F_7 = G_2 = G_5 \\
&= G_6 = G_7 = H_2 = H_3 = H_4 = H_7 = I_1 = I_3 = I_4 = I_5 = I_6 \\
&= K_1 = K_2 = K_3 = K_5 = K_7 = L_i = M_1 = M_2 = M_4 \\
&= M_6 = M_7 = 0.
\end{aligned}
\tag{9.2}
$$

Substituting these values in (8.6) we have $\mathfrak{Q}_1 = 0$[33]. In other words, the map $\mathfrak{Q}_i$ which has fewer regions (namely, 6 fewer) than M can not be colored in four colors. Hence M is not 4c. irreducible.

Bibliography

This bibliography on the map coloring problem does not pretend to be exhaustive. Only those papers are listed which seem to bear significantly on the material at hand. In particular we omit references to important recent work of P. J. Heawood, on congruences (modulo 3), and of P. Franklin, on the coloring of maps on non-orientable surfaces.

George D. Birkhoff

1. *The reducibility of maps*, Amer. J. Math. vol. 35 (1912) pp. 115–128.
2. *A determinant formula for the number of ways of coloring a map*, Ann. of Math. (2) vol. 14 (1912) pp. 42–46.
3. *On the number of ways of coloring a map*, Proceedings of the Edinburgh Mathematical Society (2) vol. 2 (1930) pp. 83–91.
4. *On the polynomial expressions for the number of ways of coloring a map*, Annali della R. Scuola Normale Superiore di Pisa (Scienze Fisiche e Matematiche) (2) vol. 3 (1934) pp. 1–19.

Philip Franklin

1. *The four-color problem*, Amer. J. Math. vol. 44 (1922) pp. 225–236.
2. *Note on the four-color problem*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 16 (1938) pp. 172–184.

P. J. Heawood

1. *Map-colour theorem*, The Quarterly Journal of Pure and Applied Mathematics vol. 24 (1890) pp. 332–338.

A. B. Kempe

1. *On the geographical problem of the four colours*, Amer. J. Math. vol. 2 (1879) pp. 193–200.

C. N. Reynolds

1. *On the problem of coloring maps in four colors*, I, Ann. of Math. (2) vol. 28 (1926) pp. 1–15.
2. *On the problem of coloring maps in four colors*, II, Ann. of Math. (2) vol. 28 (1927) pp. 477–492. This paper contains a virtually complete bibliography for the years 1921–1926 and gives a reference to a paper by Errera which contains a very complete bibliography for the still earlier papers.
3. *Circuits upon polyhedra*, Ann. of Math. (2) vol. 33 (1932) pp. 367–372.

Hassler Whitney

1. *A logical expansion in mathematics*, Bull. Amer. Math. Soc. vol. 38 (1932) pp. 572–579.
2. *The coloring of graphs*, Ann. of Math. (2) vol. 33 (1932) pp. 688–718.

[33] $O_1 = 0$ also, since O_1 is a scheme in *five* colors and we are here taking $\lambda = 4$.

3. *A theorem on graphs*, Ann. of Math. (2) vol. 32 (1931) pp. 378–390.
4. *A numerical equivalent of the four color map problem*, Monatshefte für Mathematik und Physik vol. 45 (1937) pp. 207–213.

C. E. WINN
1. *A case of coloration in the four color problem*, Amer. J. Math. vol. 59 (1937) pp. 515–528.
2. *On certain reductions in the four color problem*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 16 (1938) p. 159.
3. *On the minimum number of polygons in an irreducible map*, Amer. J. Math. vol. 62 (1940) pp. 406–416.

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